THE ELEMENTS

OF

COORDINATE GEOMETRY

BY

S. L. LONEY, M.A.,

LATE PROFESSOR OF MATHEMATICS AT THE ROYAL HOLLOWAY COLLEGE
(UNIVERSITY OF LONDON),
SOMETIME FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE

PART II. TRILINEAR COORDINATES, ETC.

MACMILLAN AND CO., LIMITED
ST MARTIN'S STREET, LONDON
1923

CONTENTS

CHAP. PAGE

I. CROSS-RATIOS 1
- Harmonic Ratio 3
- Harmonic Properties of a Complete Quadrilateral . 9
- Homographic Ranges and Pencils 14
- Coaxal Homographic Ranges 22
- Involution Ranges and Pencils 23

II. TRILINEAR COORDINATES 33
- The Straight Line 39
- Cross-Ratios 51
- Complete Quadrangles and Quadrilaterals . . 52
- Areal Coordinates 56

III. TRILINEAR COORDINATES. EQUATIONS OF THE SECOND DEGREE 60
- Tangential Equation of the General Conic . . 62
- Pole and Polar 64
- Equation to the Tangents through a given point . 66
- Director Circle of the General Conic 67
- Asymptotes of the General Conic 68
- Circles connected with the Triangle of Reference . 68
- Condition that the General Equation may represent a Circle 70
- Foci and Axes of the General Conic 79
- Circumscribed and Inscribed Conics 79
- Self-conjugate Conics 88
- Four-Point and Four-Line Conics 94
- Two Tangents and Chord of Contact 103
- Areal Coordinates 110

CHAP. PAGE

IV. MISCELLANEOUS THEOREMS 115
Pascal and Brianchon Hexagons 115
Harmonic Locus and Harmonic Envelope . . 121
Common Points and Common Tangents of two Conics 123
Constant Cross-Ratio Property of Conics . . 127
Homographic Ranges on Conics 129
Maclaurin's Theorem 135
Involution Ranges on Conics 139
Homographic Plane Figures 142
Parametric Coordinates 144

V. TANGENTIAL COORDINATES 150
Meaning of $\Sigma+\lambda\Sigma'=0$ 161
Circular Points at Infinity 162
Foci and Confocal Conics 163

VI. RECIPROCAL POLARS 169
Polar Reciprocal of one Conic with respect to another. 174
Polar Reciprocal of one Circle with respect to another 178
Reciprocation of Coaxal Circles into Confocal Conics. 184

VII. PROJECTION 188
Homographic Ranges and Pencils are Projective . 192
Fundamental proposition of Projection 193
Projection of a Conic into a Circle 194
Circular points at Infinity and Projective Generalisations 196
Four-Line Conics project into Confocal Conics and Four-Point Conics into Coaxal Conics . . . 202

VIII. INVARIANTS 206
Geometrical meaning of $\Theta=0$, $\Theta'=0$, etc. . . . 209
Condition of tangency of two Conics 215
Conics inscribed in one Quadrilateral and circumscribed to another 216
Invariant Relation between the equations to the Harmonic Locus and Harmonic Envelope . . . 219

ANSWERS 227

CHAPTER I

ANHARMONIC OR CROSS-RATIOS. HOMOGRAPHIC RANGES. INVOLUTION RANGES

[On a first reading it may be thought advisable for the student to omit from Art. 17 to the end of the Chapter.]

1. If P, Q, R, S are four points lying on a straight line, the ratio of PQ to QR divided by the ratio of PS to SR is called the **anharmonic**, or **cross-, ratio,** of the four points P, Q, R, S and is denoted by the symbol $(PQRS)$. Thus

$$(PQRS)=\frac{PQ}{QR}\div\frac{PS}{SR}=\frac{PQ}{QR}\times\frac{SR}{PS}=\frac{PQ\,.\,RS}{QR\,.\,SP}.$$

The last form is perhaps the most easy to write down, especially when, as is often the case, the four points are not in order on the straight line.

[Starting with P the four elements are PQ, QR, RS and SP in cyclical order; the first element is put in the numerator, the second in the denominator, the third in the numerator, and the fourth in the denominator.]

Such a set of points lying on a straight line is called a **range.**

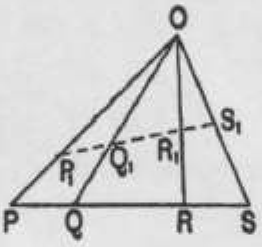
Similarly a set of straight lines meeting in a point is called a **pencil** of rays; and the point in which they meet is called their **vertex.**

The cross-ratio of a pencil of rays OP, OQ, OR, OS is

$$\frac{\sin POQ\,.\,\sin ROS}{\sin QOR\,.\,\sin SOP},$$

and is denoted by the symbol $O(PQRS)$.

2. *If a pencil OP, OQ, OR, OS is cut by any transversal in the points P, Q, R, S, the cross-ratio of the pencil is the same as that of the range $PQRS$.*



For, if p be the perpendicular from O upon the transversal $PQRS$, then $p\times PQ=2\Delta OPQ=OP\,.\,OQ\sin POQ$.

So $p \times QR = OQ \,.\, OR \sin QOR$, $p \times RS = OR \,.\, OS \sin ROS$, and $p \times SP = OS \,.\, OP \sin SOP$.

Hence, on substituting and cancelling like quantities, we have

$$[PQRS] = \frac{PQ \,.\, RS}{QR \,.\, SP} = \frac{\sin POQ}{\sin QOR} \cdot \frac{\sin ROS}{\sin SOP} = O\,[PQRS].$$

If any other transversal cuts the rays in P_1, Q_1, R_1, S_1 it follows that $[P_1Q_1R_1S_1] = [PQRS]$.

For

$$[P_1Q_1R_1S_1] = O\,[P_1Q_1R_1S_1] = O\,[PQRS] = [PQRS].$$

3. When four points such as P_1, Q_1, R_1, S_1 are obtained from P, Q, R, S, or *vice versa*, by joining one range of four points to a vertex O, and obtaining the second range by means of the intersections of the four joining lines with any transversal, then one range of the points is said to be the projection of the other range.

Thus the cross-ratio of a range of points is the same as that of their projections, or, as it is generally expressed, the cross-ratio of a range of four points is unaltered by projection. As will be seen in a later chapter this is a most important result. [Arts. 199–201.]

4. *Cross-ratio of the pencil formed by the four straight lines* $y = m_1x$, $y = m_2x$, $y = m_3x$ *and* $y = m_4x$.

Let a straight line parallel to the axis of y meet the four straight lines in the points P, Q, R, S and the axis of x in N. Let $ON = h$, where O is the origin. Then

$$NP = m_1h;\; NQ = m_2h;\; NR = m_3h;\; NS = m_4h;$$

$$\therefore\; PQ = (m_2 - m_1)\,h;\; QR = (m_3 - m_2)\,h;\; RS = (m_4 - m_3)\,h$$

and

$$SP = (m_1 - m_4)\,h.$$

Hence the cross-ratio of the pencil OP, OQ, OR, OS

$$= \text{that of the range } P, Q, R, S = [PQRS]$$

$$= \frac{PQ \,.\, RS}{QR \,.\, SP} = \frac{(m_2 - m_1)\,(m_4 - m_3)}{(m_3 - m_2)\,(m_1 - m_4)}.$$

This result is easily seen to be true whether the axes are rectangular or oblique.

5. *Cross-ratio of the pencil formed by the straight lines whose equations are*

$$y=0, \quad y=m_1x, \quad x=0, \quad y=m_2x.$$

In this case the third straight line cuts the straight line parallel to the axis of y, *i.e.* parallel to itself, at an infinite distance.

Also the points N and P of the previous article coincide; hence in this case we have

$$PQ=m_1h, \quad PR=\infty, \quad \text{and} \quad PS=m_2h,$$

and thus

$$[PQRS]=\frac{PQ\,.\,RS}{QR\,.\,SP}=\frac{m_1h}{\infty-m_1h}\cdot\frac{m_2h-\infty}{-m_2h}=\frac{m_1}{m_2}.$$

6. Harmonic Ratio. If the cross-ratio of four points is equal to -1, the ratio is said to be harmonic.

In this case, from the definition of Art. 1,

$$\frac{PQ}{QR}=-\frac{PS}{SR}=\frac{PS}{RS}.$$

$$\therefore\ PQ(PS-PR)=PS(PR-PQ).$$

$$\therefore\ PR\,.\,PQ+PS\,.\,PR=2PS\,.\,PQ.$$

$$\therefore\ \frac{1}{PQ}+\frac{1}{PS}=\frac{2}{PR},$$

i.e. PQ, PR and PS are in harmonic progression.

If Q is the middle point of PR, then S is at infinity.

If O be the middle point of PR, then $OP^2=OQ\,.\,OS$.

For if

$$PQ=q, \quad PR=r \quad \text{and} \quad PS=s, \quad \text{then} \quad r=\frac{2qs}{q+s}.$$

$$\therefore\ OQ=q-\frac{r}{2}=\frac{q^2}{q+s}, \quad \text{and} \quad OS=s-\frac{r}{2}=\frac{s^2}{q+s}.$$

$$\therefore\ OQ\,.\,OS=\frac{q^2s^2}{(q+s)^2}=\frac{r^2}{4}=OP^2.$$

The points Q and S are said to be **harmonically conjugate** with respect to the points P and R; this is also expressed by saying that they divide the straight line PR harmonically; the points Q and S are said to be conjugate points as also the points P and R.

7. Similarly, if a pencil of four lines is such that its cross-ratio is equal to -1, it is called a harmonic pencil, and two of the lines divide the angle between the other two harmonically.

In the proposition of Part I, Art. 401, we have an example of a harmonic range, and the proposition may be expressed in the form

$$(OPRP') = -1.$$

8. If x_1, x_2, x_3, x_4 are the distances of four points P, Q, R, S from any fixed origin O lying on the straight line on which the four points lie, then

$$(PQRS) = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_2)(x_1 - x_4)}.$$

This quantity is equal to -1, if

$$(x_2 - x_1)(x_4 - x_3) + (x_3 - x_2)(x_1 - x_4) = 0,$$

i.e. if $$\mathbf{(x_1 + x_3)(x_2 + x_4) = 2x_1x_3 + 2x_2x_4},$$

which is therefore the condition that the four points should form a harmonic range.

The cross-ratio of the four points is equal to unity if

$$(x_2 - x_1)(x_4 - x_3) = (x_3 - x_2)(x_1 - x_4),$$

i.e. if $$(x_1 - x_3)(x_2 - x_4) = 0,$$

and then $$x_1 = x_3 \text{ or } x_2 = x_4,$$

so that either the points P and R coincide or the points Q and S coincide in this case.

Similarly the straight lines $y = m_1x$, $y = m_2x$, $y = m_3x$, and $y = m_4x$ form a harmonic pencil if

$$\mathbf{(m_1 + m_3)(m_2 + m_4) = 2m_1m_3 + 2m_2m_4}.$$

Just as in Art. 5 it follows that the straight lines $y = 0$, $y = px$, $x = 0$ and $y = -px$ form a harmonic pencil.

9. *To find the condition that the straight lines given by the equations $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ may be harmonically conjugate.*

Let the first pair of straight lines be $y = m_1x$ and $y = m_2x$, so that $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$.

Similarly, let the second pair be $y = m_2 x$ and $y = m_4 x$, so that $m_2 + m_4 = -\frac{2h'}{b'}$ and $m_2 m_4 = \frac{a'}{b'}$.

By Art. 4 we then have

$$\frac{(m_2 - m_1)(m_4 - m_3)}{(m_3 - m_2)(m_1 - m_4)} = -1,$$

i.e.
$$(m_1 + m_3)(m_2 + m_4) = 2m_1 m_3 + 2m_2 m_4.$$

Hence
$$-\frac{2h}{b} \times \left(-\frac{2h'}{b'}\right) = \frac{2a}{b} + \frac{2a'}{b'}.$$

$$\therefore\ ab' + a'b = 2hh'$$

is the required condition.

Cor. 1. The pair of straight lines $y^2 - p^2x^2 = 0$ and the pair given by $xy = 0$ are harmonically conjugate, *i.e.* any pair of straight lines and the bisectors of the angles between them form a harmonic pencil.

Cor. 2. The imaginary straight lines $x^2 + y^2 = 0$ and the axes $xy = 0$ also satisfy the condition of this article. Hence any pair of perpendicular straight lines divide harmonically the lines joining their intersection to the circular points at infinity (Part I, Art. 392).

10. Just as in the previous article a pair of points whose distances from a fixed origin are given by $ax^2 + 2hx + b = 0$ divide the straight line joining the pair of points given by $a'x^2 + 2h'x + b' = 0$ harmonically if $ab' + a'b = 2hh'$.

11. *To find the pairs of lines passing through the origin which are harmonically conjugate both with the pair of lines given by* $ax^2 + 2hxy + by^2 = 0$ *and with the pair given by* $a_1x^2 + 2h_1xy + b_1y^2 = 0$.

To shew also that the lines so found are harmonically conjugate with respect to all the pairs given by

$$ax^2 + 2hxy + by^2 + \lambda(a_1x^2 + 2h_1xy + b_1y^2) = 0,$$

where λ *is any constant.*

Let the required pair be

$$a'x^2 + 2h'xy + b'y^2 = 0 \ \ldots\ldots\ldots\ldots\ldots(1).$$

Then, by the previous article, we have

$$a'b - 2h'h + b'a = 0 \quad \text{...............(2)},$$

and

$$a'b_1 - 2h'h_1 + b'a_1 = 0 \quad \text{...............(3)}.$$

Solving, we have

$$\frac{a'}{ah_1 - a_1 h} = \frac{2h'}{ab_1 - a_1 b} = \frac{b'}{hb_1 - h_1 b}.$$

Also (2) and (3) give, for all values of λ,

$$a'(b + \lambda b_1) - 2h'(h + \lambda h_1) + b'(a + \lambda a_1) = 0.$$

But this is the condition that (1) should be harmonically conjugate with the pair

$$(a + \lambda a_1) x^2 + 2(h + \lambda h_1) xy + (b + \lambda b_1) y^2 = 0.$$

12. *If four points lie on a straight line, their polars with regard to any conic meet in a point, and the cross-ratio of the four points is equal to that of the pencil formed by their polars.*

Take the straight line as the axis of x, and let the distances of the four points P_1, P_2, P_3, P_4 from the origin be x_1, x_2, x_3, x_4. The polar of the point P_1, *i.e.* $(x_1, 0)$, with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is

$$x(ax_1 + g) + y(hx_1 + f) + gx_1 + c = 0 \quad \text{......(1)}.$$

For all values of x_1 this passes through a given point, viz. the pole of the axis of x.

Now (1) is inclined to the axis at an angle $\tan^{-1} m_1$, where

$$m_1 = -\frac{hx_1 + f}{ax_1 + g}.$$

So for the polars of P_2, P_3, P_4.

Hence the cross-ratio of the pencil of polars

$$= \frac{(m_2 - m_1)(m_4 - m_3)}{(m_3 - m_2)(m_1 - m_4)} = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_2)(x_1 - x_4)},$$

on substitution and reduction,

= the cross-ratio of the four given points.

13. From four points P, Q, R and S, lying on a straight line, we can obtain different cross-ratios according to the order in which we take the points. Four points can be permuted amongst themselves in $\underline{|4}$, *i.e.* 24, ways, but it is easily seen that they only give rise to six different cross-ratios. For

$$[PQRS]=[QPSR]=[RSPQ]=[SRQP] \quad ...(1),$$
$$[PQSR]=[QPRS]=[RSQP]=[SRPQ] \quad ...(2),$$
$$[PRQS]=[QSPR]=[RPSQ]=[SQRP] \quad ...(3),$$
$$[PRSQ]=[QSRP]=[RPQS]=[SQPR] \quad ...(4),$$
$$[PSQR]=[QRPS]=[RQSP]=[SPRQ] \quad ...(5),$$
$$[PSRQ]=[QRSP]=[RQPS]=[SPQR] \quad ...(6),$$

as may be verified by the definition of Art. 1.

The second, third, and fourth cross-ratios in either of these six rows may be obtained from the first by interchanging any two letters in it and then interchanging the other two. Thus, if in $[PQRS]$ we interchange P and S, and then interchange Q and R, we obtain $[SRQP]$, which is the last ratio in (1). We thus have six different cross-ratios, viz. the first in each of the above rows.

These again are not independent. For if P, Q, R, S are any four points in a straight line, in any order, we always have

$$PQ\,.\,RS+PR\,.\,SQ+PS\,.\,QR=0 \quad ....(7),$$

where careful attention is paid to the signs of the quantities involved.

For let the distances P, Q, R, S from a fixed point in the straight line be p, q, r, s so that

$$PQ=OQ-OP=q-p;\ RS=OS-OR=s-r, \text{ etc.},$$

then the left-hand member of (7)

$$=(q-p)(s-r)+(r-p)(q-s)+(s-p)(r-q)=0.$$

Dividing (7) by $PS\,.\,QR$, we have

$$\frac{PQ\,.\,RS}{PS\,.\,QR}+\frac{PR\,.\,SQ}{PS\,.\,QR}+1=0,$$

$$\therefore\ \frac{PR}{RQ}\cdot\frac{QS}{SP}=1-\frac{PQ}{QR}\cdot\frac{RS}{SP}.$$

Hence, if we put $[PQRS]=\lambda$, this gives

$$[PRQS]=1-\lambda.$$

Again, dividing (7) by $PQ\,.\,RS$, we have

$$1+\frac{PR\,.\,SQ}{PQ\,.\,RS}+\frac{PS\,.\,QR}{PQ\,.\,RS}=0,$$

$$\therefore\ [PRSQ]=\frac{PR\,.\,SQ}{RS\,.\,QP}=1-\frac{QR\,.\,SP}{RS\,.\,PQ}=1-\frac{1}{\lambda}=\frac{\lambda-1}{\lambda}.$$

Further, if we interchange the first and third letters or the second and fourth points of a range, we easily see that we invert a cross-ratio. Thus

$$[PSRQ]=\frac{1}{[PQRS]}=\frac{1}{\lambda},\quad [PSQR]=\frac{1}{[PRQS]}=\frac{1}{1-\lambda},$$

and $$[PQSR]=\frac{1}{[PRSQ]}=1\div\frac{\lambda-1}{\lambda}=\frac{\lambda}{\lambda-1}.$$

Thus finally the six cross-ratios (1), (2), (3), (4), (5), and (6) become

$$\lambda,\ \frac{\lambda}{\lambda-1},\ 1-\lambda,\ \frac{\lambda-1}{\lambda},\ \frac{1}{1-\lambda},\ \text{and}\ \frac{1}{\lambda}.$$

14. The 24 cross-ratios that can be obtained by taking the 4 points P, Q, R, S in any order have thus been expressed in terms of any one of them.

On examination it will be seen that four of the cross-ratios are always positive, and the other two always negative.

In the case when P, Q, R, S form a harmonic range, if we put $\lambda=-1$, the other cross-ratios become $\frac{1}{2}$, 2, 2, $\frac{1}{2}$, and -1, respectively, *i.e.* the 24 cross-ratios reduce to the values -1, $\frac{1}{2}$ and 2. Hence a range of four points is harmonic, if their cross-ratio, when the points are taken in any order, reduces to either of the values -1, $\frac{1}{2}$, or 2.

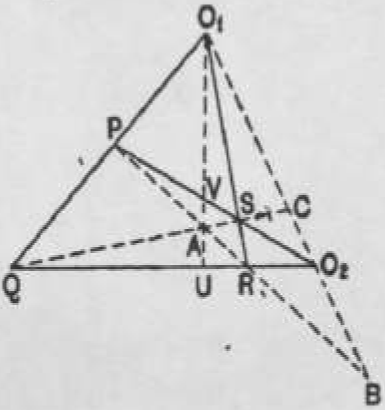
Similarly, if either of the cross-ratios is equal to 1, 0, or ∞, then two of the four points coincide.

If we put $\lambda=-\tan^2\theta$, the six possible values of the cross-ratios in the previous article can be written in the form $\sin^2\theta$, $\cos^2\theta$, $-\tan^2\theta$, $-\cot^2\theta$, $\operatorname{cosec}^2\theta$ and $\sec^2\theta$. If circles be described on PR and QS which meet in T, and if O, O_1 be their centres, it can be shewn that θ is one-half the angle OTO_1. For if $PQ=b$, $PR=c$, and $PS=d$, the triangle OTO_1 gives $\cos OTO_1=\dfrac{c\,(d+b)-2db}{c\,(d-b)}$, and hence

$$\tan^2\frac{OTO_1}{2}=\frac{b\,(d-c)}{d\,(c-b)}=-\frac{PQ\,.\,RS}{QR\,.\,SP}=-(PQRS).$$

15. *To shew that each diagonal of a complete quadrilateral is divided harmonically by the other two diagonals.*

Let $PQRS$ be a quadrilateral.



Produce the sides to meet in O_1 and O_2.

Take QR and QP as the axes of x and y. Let the equation of PR be

$$l_1x + m_1y - 1 = 0 \qquad (1),$$

and that of O_1O_2 be

$$l_2x + m_2y - 1 = 0 \qquad (2).$$

Then the equation of RO_1 is

$$l_1x + m_2y - 1 = 0 \qquad (3),$$

and that of O_2P is

$$l_2x + m_1y - 1 = 0 \qquad (4).$$

By subtracting (2) from (1), we see that the equation of QB is

$$(l_1 - l_2)x + (m_1 - m_2)y = 0.$$

Also, subtracting (4) from (3), we have as the equation of QS

$$(l_1 - l_2)x - (m_1 - m_2)y = 0.$$

Hence, by Art. 8, QB, QR, QA and QP form a harmonic pencil, so that $(BRAP)$ and (BO_2CO_1) are harmonic ranges.

Similarly it may be shewn that $[QASC]$ is a harmonic range.

Aliter. This proposition may also be deduced from Ceva's and Menelaus' Theorems.

Let O_1A meet QR in U.

From the triangle O_1PR since O_1U, QS, and RP meet in a point, Ceva's Theorem gives

$$QU\,.\,RS\,.\,O_1P = UR\,.\,SO_1\,.\,PQ.$$

From the same triangle, since PSO_2 is a transversal, Menelaus' Theorem gives

$$QO_2\,.\,RS\,.\,O_1P = -\,O_2R\,.\,SO_1\,.\,PQ.$$

Hence, by division,

$$\frac{QU}{UR} = -\frac{QO_2}{O_2R}, \text{ i.e. } \frac{QU}{UR}\cdot\frac{RO_2}{O_2Q} = -1.$$

$\therefore$ $[QURO_2]$ is harmonic.

$\therefore$ $O_1[QURO_2]$ is a harmonic pencil.

Hence, by Art. 2, $[QASC] = -1$ and $[PARB] = -1$.

Also, since the pencil $Q[PARB] = -1$, therefore the transversal BO_1 gives

$$[O_1CO_2B] = -1.$$

Cor. The pencils at the three vertices A, O_1, O_2, viz. $A(PO_1SO_2)$, $O_1(PASO_2)$, and $O_2(QAPO_1)$, are all harmonic.

16. *If four points A, B, C, D on a conic be given, the cross-ratio of the pencil $P(ABCD)$ is constant for all positions of P on the conic.*

This follows from the final result of Part I, Art. 383. For the perpendicular from P upon AB

$$= \frac{2\triangle PAB}{AB} = \frac{PA\,.\,PB \sin APB}{AB},$$

and so for the other three perpendiculars.

$$\therefore \left(\frac{PA\,.\,PB \sin APB}{AB} \times \frac{PC\,.\,PD \sin CPD}{CD}\right) \div \left(\frac{PB\,.\,PC}{BC}\sin BPC \times \frac{PD\,.\,PA}{DA}\sin DPA\right)$$

is constant, *i.e.*

$$\frac{\sin APB}{\sin BPC}\cdot\frac{\sin CPD}{\sin DPA} = k\,\frac{AB\,.\,CD}{BC\,.\,DA},$$

where k is a constant, *i.e.* $P[ABCD]$ is constant.

Ex. Shew that the cross-ratio of the pencil subtended at any point P of an ellipse by the four points whose eccentric angles are $2\alpha_1$, $2\alpha_2$, $2\alpha_3$, $2\alpha_4$ is

$$\frac{\sin(\alpha_2-\alpha_1)\sin(\alpha_4-\alpha_3)}{\sin(\alpha_3-\alpha_2)\sin(\alpha_1-\alpha_4)}.$$

[Take P at an end of the major axis of the ellipse, so that

$$\tan\theta_1=\frac{b\sin 2\alpha_1}{a+a\cos 2\alpha_1}=\frac{b}{a}\tan\alpha_1, \text{ etc.]}$$

EXAMPLES.

1. *Shew that the locus of a point the tangents from which to the circle $x^2+y^2-2g_1x+c=0$ harmonically divide the tangents from it to the circle $x^2+y^2-2g_2x+c=0$ is*

$$x^2(2c-g_1^2-g_2^2)+2y^2(c-g_1g_2)-2x(g_1+g_2)(c-g_1g_2)$$
$$+2c^2-c(g_1^2+g_2^2)=0.$$

By Part I, Art. 389, parallels drawn through the origin to the tangents from (x', y') to the first circle are given by

$$x^2(y'^2+c-g_1^2)+y^2(x'^2-2g_1x'+c)-2xyy'(x'-g_1)=0,$$

and similarly for the second circle they are given by

$$x^2(y'^2+c-g_2^2)+y^2(x'^2-2g_2x'+c)-2xyy'(x'-g_2)=0.$$

By Art. 9, these are harmonic if

$$(y'^2+c-g_1^2)(x'^2-2g_2x'+c)+(y'^2+c-g_2^2)(x'^2-2g_1x'+c)$$
$$=2y'^2(x'-g_1)(x'-g_2).$$

Hence, on reduction, the locus of (x', y') is

$$x^2(2c-g_1^2-g_2^2)+2y^2(c-g_1g_2)-2x(g_1+g_2)(c-g_1g_2)$$
$$+2c^2-c(g_1^2+g_2^2)=0.$$

If the two given circles cut orthogonally, then

$$(g_1-g_2)^2=g_1^2-c+g_2^2-c, \text{ i.e. } c=g_1g_2,$$

and the locus reduces to

$$x^2=-c,$$

i.e. a pair of straight lines, which are real since the circles intersect in real points, and therefore c is negative.

2. *A straight line is drawn so that its intersections with the circle $x^2+y^2-2g_1x+c=0$ are always harmonically conjugate to its intersections with the circle $x^2+y^2-2g_2x+c=0$; shew that its envelope is a conic whose foci are the centres of the circles, and that this envelope reduces to the two centres only if the circles cut orthogonally.*

The lines joining the origin to the intersections of the straight line $lx+my+n=0$ with the first circle are

$$n^2(x^2+y^2)+2g_1nx(lx+my)+c(lx+my)^2=0,$$

i.e. $$x^2(n^2+2g_1ln+cl^2)+2xym(g_1n+lc)+y^2(n^2+cm^2)=0.$$

Similarly for the second circle.

These are harmonic if (Art. 9)

$$(n^2+cm^2)[2n^2+2(g_1+g_2)ln+2cl^2]=2m^2(g_1n+lc)(g_2n+lc),$$

i.e. if $$m^2(c-g_1g_2)+n^2+(g_1+g_2)ln+cl^2=0,$$

i.e. if $$\frac{(lg_1+n)(lg_2+n)}{l^2+m^2}=g_1g_2-c \quad\text{.....................(i)},$$

i.e. if the product of the perpendicular from the centres of the two circles upon the straight line is constant. Its envelope is thus a conic, with these two centres as foci, and its semi-minor axis equal to $\sqrt{g_1g_2-c}$.

If the circles cut orthogonally, then $g_1g_2=c$, and (i) gives either $lg_1+n=0$, or $lg_2+n=0$, *i.e.* the straight line passes through one or other of the centres of the two circles. This is clear geometrically; for if two circles cut orthogonally any line through the centre O of one cuts that circle in P_1 and P_2 and the other in Q_1, Q_2 such that $OQ_1 . OQ_2=(\text{rad. of first circle})^2=OP_1^2=OP_2^2$, and thus $(P_1Q_1P_2Q_2)$ is harmonic (Art. 6).

3. *Shew that one of the cross-ratios, λ, of the pencil in which the pair of lines $ax^2+2hxy+by^2=0$ are conjugates, and the other pair of conjugates is the lines $a'x^2+2h'xy+b'y^2=0$ is given by*

$$\left(\frac{\lambda-1}{\lambda+1}\right)^2=\frac{4(h^2-ab)(h'^2-a'b')}{(ab'+a'b-2hh')^2}.$$

Let the first pair be $(y-p_1x)(y-p_3x)=0$ and the second pair $(y-p_2x)(y-p_4x)=0$, so that

$$p_1+p_3=-\frac{2h}{b}, \qquad p_1p_3=\frac{a}{b},$$

$$p_2+p_4=-\frac{2h'}{b'}, \text{ and } p_2p_4=\frac{a'}{b'}.$$

Then $$\lambda=\frac{(p_2-p_1)(p_4-p_3)}{(p_3-p_2)(p_1-p_4)},$$

and hence $$\frac{\lambda-1}{\lambda+1}=\frac{(p_1-p_3)(p_2-p_4)}{2p_1p_3+2p_2p_4-(p_1+p_3)(p_2+p_4)}$$

$$=\frac{2\sqrt{h^2-ab}\sqrt{h'^2-a'b'}}{ab'+a'b-2hh'},$$

from the above relations.

The cross-ratio is harmonic if $\lambda=-1$, and then $ab'+a'b=2hh'$ as in Art. 9.

4. *Shew that the points whose abscissae are the roots of the quartic*

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

form a harmonic range if

$$(bc-ad)^2=(b^2-ac)(c^2-ae).$$

If the roots are x_1, x_2, x_3, x_4, the points give a harmonic range (Art. 8) if

$$(x_1+x_3)(x_2+x_4)=2(x_1x_3+x_2x_4) \quad\text{..................(1).}$$

But

$$-\frac{4b}{a}=(x_1+x_3)+(x_2+x_4),$$

$$\frac{6c}{a}=x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4,$$

i.e. by (1), $$\frac{4c}{a}=2(x_1x_3+x_2x_4)=(x_1+x_3)(x_2+x_4),$$

$$-\frac{4d}{a}=x_1x_3(x_2+x_4)+x_2x_4(x_1+x_3),$$

and $$\frac{e}{a}=x_1x_3\times x_2x_4.$$

$$\begin{aligned}\therefore\ 16\frac{b^2-ac}{a^2}&=[(x_1+x_3)+(x_2+x_4)]^2-4(x_1+x_3)(x_2+x_4)\\&=[(x_1+x_3)-(x_2+x_4)]^2,\end{aligned}$$

$$16\frac{c^2-ae}{a^2}=4(x_1x_3-x_2x_4)^2,$$

and $$\begin{aligned}16\frac{bc-ad}{a^2}&=-2[(x_1+x_3)+(x_2+x_4)][x_1x_3+x_2x_4]\\&\qquad+4x_1x_3(x_2+x_4)+4x_2x_4(x_1+x_3)\\&=-2[(x_1+x_3)-(x_2+x_4)][x_1x_3-x_2x_4].\end{aligned}$$

$$\therefore\ (bc-ad)^2=(b^2-ac)(c^2-ae).$$

This may be written in the form

$$ace+2bcd-ad^2-eb^2-c^3=\begin{vmatrix}a, & b, & c\\ b, & c, & d\\ c, & d, & e\end{vmatrix}=0.$$

5. O, A, B, C are four collinear points, A' is the harmonic conjugate of O with respect to B and C, Q is the harmonic conjugate of O with respect to A and A', and P is the harmonic conjugate of A with respect to Q and A'; show that

$$\frac{3}{OP}=\frac{1}{OA}+\frac{1}{OB}+\frac{1}{OC}.$$

[Let $OA=a$, $OB=b$, $OC=c$, $OA'=a'$, $OQ=q$ and $OP=p$. Then

$$\frac{2}{a'}=\frac{1}{b}+\frac{1}{c};\ \frac{2}{q}=\frac{1}{a}+\frac{1}{a'};\ \text{and}\ \frac{2}{p-a}=\frac{1}{q-a}+\frac{1}{a'-a}.$$

Hence $$q-a=\frac{a(a'-a)}{a'+a};\ p=\frac{3aa'}{a'+2a}=\frac{3abc}{bc+ca+ab},\ \text{etc.}]$$

6. A, B, C are collinear points and X is the fourth harmonic of A with respect to B and C; Y is the fourth harmonic of B with respect to C and A; and Z is the fourth harmonic of C with respect to A and B; prove that

$$\frac{1}{AX}+\frac{1}{BY}+\frac{1}{CZ}=0.$$

7. If B_0, B_2 are harmonic conjugates to B_1, C, and B_1, B_3 to B_2, C, and B_4, B_2 to B_3, C, show that B_0, B_4 are harmonic conjugates to B_2, C.

[Take any vertex O and estimate the pencils thus obtained on a transversal parallel to OC, so that the point C_1 is at infinity, *i.e.* in the language of a later chapter project C to infinity.]

8. Two sets of four collinear points are such that the points of one set are at distances $2, \frac{1}{2}, -1, 1$ from a fixed point on the line, and those of the other set at distances $-3, \frac{3}{2}, -4, -\frac{39}{7}$ from a fixed point on the other line. Shew that two ranges of four points can be formed from the two sets which have the same cross-ratios.

[The cross-ratios of the first set are, as in Art. 13, equal to $2, 2, -1, \frac{1}{2}, -1, \frac{1}{2}$ and those of the second are $\frac{1}{2}, -1, \frac{1}{2}, -1, 2, 2$. Hence, with the notation of Art. 13, the ranges $PQRS$ and $P_1S_1R_1Q_1$ have the same cross-ratios.]

9. Shew that the asymptotes and any pair of conjugate diameters of a conic form a harmonic pencil.

Homographic Ranges.

17. A range of points on one line is said to have a one-one correspondence with a range of points on a second line [which line may coincide with the first line] if to each point of the first range there corresponds one, and one only, point on the second range, and if also to each point of the second range there corresponds one, and one only, point of the first range.

Let the distance of any point of the first range from a fixed point on it be x, and the distance of the corresponding point of the second range from a fixed point on it be x'. When the lines on which the ranges lie intersect, the fixed point for each may be taken to be the common point of intersection.

Then, assuming the relation between x and x' to be algebraic, it must be of the form

$$Axx' + Bx + Cx' + D = 0 \;\ldots\ldots\ldots\ldots\ldots(1),$$

where A, B, C and D are constants. For if in this relation we had a term of the form Ex'^2, then, although for each value of x' there would be only one of x, yet for a given value of x we should have a quadratic equation for x', and therefore more than one value of x'. Similarly it would be seen that any other terms, besides the four in the relation (1), would be impossible.

When such a relation as (1) holds, the point on the second range is uniquely determined when the corresponding point of the first range is given. The two ranges are said to be **homographic**.

Similarly **homographic pencils** are defined; if corresponding rays are inclined to any fixed lines at angles $\tan^{-1} m$ and $\tan^{-1} m'$, then

$$Amm' + Bm + Cm' + D = 0.$$

18. *The cross-ratio of any four points of the first range is equal to that of the corresponding four points of the second range.*

For let the distances of the four points of the first range from the origin be x_1, x_2, x_3, x_4, and let those of the second range be x_1', x_2', x_3', x_4', so that we have four relations such as

$$Ax_1x_1' + Bx_1 + Cx_1' + D = 0.$$

Hence $$x_1' = -\frac{Bx_1 + D}{Ax_1 + C}, \text{ and } x_2' = -\frac{Bx_2 + D}{Ax_2 + C},$$

and hence $$x_2' - x_1' = \frac{(AD - BC)(x_2 - x_1)}{(Ax_1 + C)(Ax_2 + C)},$$

and similarly for $x_3' - x_2'$, $x_4' - x_3'$ and $x_1' - x_4'$.

Now, by Art. 8, the cross-ratio of the second set of points

$$= \frac{(x_2' - x_1')(x_4' - x_3')}{(x_3' - x_2')(x_1' - x_4')} = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_2)(x_1 - x_4)}$$

(on making the substitutions such as the above)

= the cross-ratio of the four points of the first range.

Homographic ranges are thus ranges whose cross-ratios are equal.

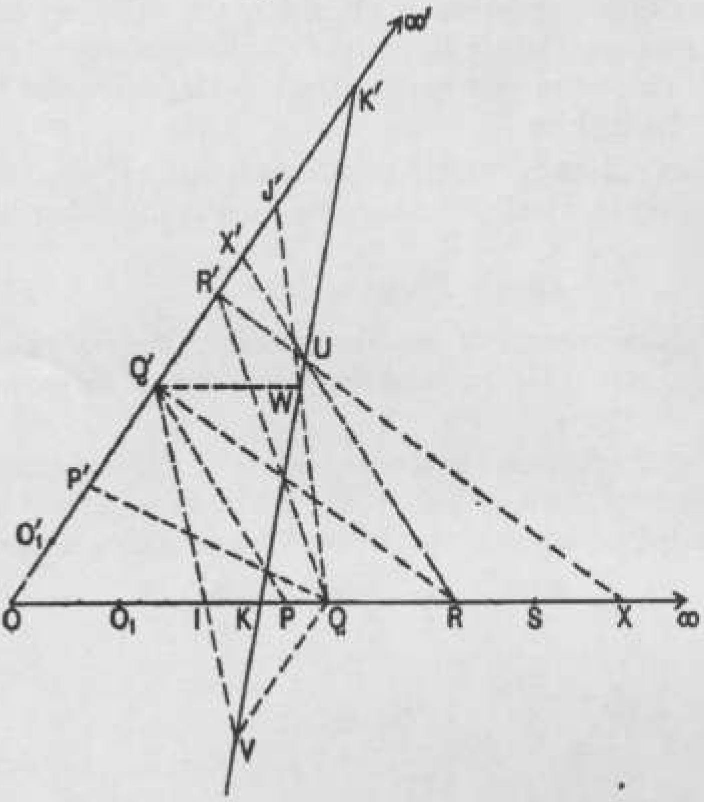
19. *Points on either ranges corresponding to the points at an infinite distance on the other range.*

The relation connecting *any* pair of points such as P and P', or Q and Q', etc., being

$$Axx' + Bx + Cx' + D = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

i.e. $$x' = -\frac{Bx + D}{Ax + C}, \text{ or } x = -\frac{Cx' + D}{Ax' + B},$$

then to any point at infinity on one range corresponds a point on the other range.



If $x=\infty$, then

$$x' = -\frac{B+\dfrac{D}{x}}{A+\dfrac{C}{x}} = -\frac{B}{A}.$$

Let this value of x' give the point J', so that $O_1'J' = -\frac{B}{A}$.

If $x'=\infty$, then similarly $x=-\frac{C}{A}$, giving the point I, so that $O_1I=-\frac{C}{A}$.

If we now take I and J' as the origins of measurement, *i.e.* if we put

$$x = O_1I+\xi = -\frac{C}{A}+\xi,$$

and $$x' = O_1'J' + \xi' = -\frac{B}{A} + \xi',$$

then (1) becomes

$$\xi\xi' = \frac{BC - AD}{A^2}.$$

Hence, if P, P' be any pair of corresponding points, then $IP \,.\, J'P' =$ constant.

Conversely if $IP \,.\, J'P'$ is constant where I, J' are fixed points on two ranges, and P, P' are variable corresponding points, then P, P' describe ranges which are homographic.

These points I and J' are called **Vanishing Points,** so that a vanishing point on one range is the point corresponding to the point at infinity on the other range.

20. If $C = 0$, the origin O_1 for x coincides with I.

If $B = 0$, the origin O_1' for x' is J'.

If both $B = 0$ and $C = 0$, the origins O_1 and O_1' for x and x' are I and J' respectively.

If $D = 0$, the origins O_1 and O_1' are corresponding points.

21. It may be noted that in the fundamental relation we must not have $\frac{A}{B} = \frac{C}{D}$; for if each of these quantities were equal to λ, so that $A = B\lambda$ and $C = D\lambda$, the relation would become $(Bx + D)(\lambda x' + 1) = 0$. In this case we should have $x = -\frac{D}{B}$ with any value for x', or $x' = -\frac{1}{\lambda} = -\frac{D}{C}$ with any value for x, and hence a definite point of one range would correspond to any point of the other, and we should not have two ranges of points corresponding one to one.

22. In order to give definitely two homographic ranges three pairs of corresponding points must be given. For the relation (1) of Art. 19 contains three constants $\frac{A}{D}$, $\frac{B}{D}$ and $\frac{C}{D}$; hence, when three pairs of corresponding values of x and x' are given, we have three simple equations to determine these three constants.

If the three pairs of corresponding points be at distances a and a', b and b', c and c' from their origins, we have

$$Aaa' + Ba + Ca' + D = 0,$$

$$Abb' + Bb + Cb' + D = 0,$$

and

$$Acc' + Bc + Cc' + D = 0.$$

Hence

$$\frac{A}{\begin{vmatrix} 1, & a, & a' \\ 1, & b, & b' \\ 1, & c, & c' \end{vmatrix}} = \frac{B}{\begin{vmatrix} aa', & 1, & a' \\ bb', & 1, & b' \\ cc', & 1, & c' \end{vmatrix}} = \frac{C}{\begin{vmatrix} aa', & a, & 1 \\ bb', & b, & 1 \\ cc', & c, & 1 \end{vmatrix}} = \frac{-D}{\begin{vmatrix} aa', & a, & a' \\ bb', & b, & b' \\ cc', & c, & c' \end{vmatrix}}.$$

23. *If there are any number of points P, Q, R, S, ... on one straight line, and a homographic system P', Q', R', S',... on another straight line, the locus of the points of intersection of all such pairs as PQ', $P'Q$ is a straight line, and the envelope of the lines joining corresponding points is a conic.*

Take the two straight lines as the axes of x and y so that the common origin for both ranges is now O, their point of intersection (Fig., Art. 19). The distances x_1 and y_1, x_2 and y_2, of any two pairs of corresponding points P and P', Q and Q', are given by the equation

$$Ax_1y_1 + Bx_1 + Cy_1 + D = 0 \quad \text{............(1)},$$

i.e.

$$y_1 = -\frac{Bx_1 + D}{Ax_1 + C}$$

and

$$y_2 = -\frac{Bx_2 + D}{Ax_2 + C}.$$

The equation to PQ' is

$$\frac{x}{x_1} + \frac{y}{y_2} = 1,$$

i.e.

$$x(Bx_2 + D) - yx_1(Ax_2 + C) = x_1(Bx_2 + D).$$

So that of $P'Q$ is

$$x(Bx_1 + D) - yx_2(Ax_1 + C) = x_2(Bx_1 + D).$$

By subtraction, a line through their point of intersection is

$$Bx(x_2 - x_1) + Cy(x_2 - x_1) = -D(x_2 - x_1),$$

i.e.

$$Bx + Cy + D = 0 \quad \text{.................(2)}.$$

This is therefore the required locus. It passes through the points $\left(-\frac{D}{B}, 0\right)$ and $\left(0, -\frac{D}{C}\right)$. These are the points, K' and K, on the two lines which are respectively the points corresponding to the origin O, when the latter is looked upon as belonging successively to the two ranges. For when $x_1 = 0$, $y_1 = -\frac{D}{C}$, and when $y_1 = 0$, $x_1 = -\frac{D}{B}$.

The line (2) which is the locus of intersection of cross lines is called the **Homographic**, or **Cross-, Axis** of the two ranges.

The equation to PP' is $\frac{x}{x_1} + \frac{y}{y_1} = 1$, *i.e.* $lx + my = 1$, where $Dlm + Cl + Bm + A = 0$, on putting $l = \frac{1}{x_1}$ and $m = \frac{1}{y_1}$ in (1).

By Art. 437, Part I, the envelope of PP' is

$$B^2x^2 + C^2y^2 + D^2 + 2yCD + 2xBD + 2xy(2AD - BC) = 0,$$

i.e.
$$(Bx + Cy + D)^2 = 4xy(BC - AD),$$

i.e. a conic touching the two ranges at the points, K and K', where the axis of homography meets them.

If $BC = AD$, this conic reduces to a pair of coincident straight lines $(Bx + Cy + D)^2 = 0$. This is the excepted case of Art. 21.

It also reduces to a pair of coincident straight lines

$$(Bx - Cy)^2 = 0,$$

if $D = 0$, and then the point of intersection of the two ranges is a point corresponding to itself in each range, and is therefore a common point of the two ranges.

In this case the relation (1) gives $y_1 = -\frac{Bx_1}{Ax_1 + C}$ and the equation to PP' becomes $Bx - y(Ax_1 + C) = Bx_1$, which, for all values of x_1, passes through the fixed point $\left(-\frac{C}{A}, -\frac{B}{A}\right)$. When, as here, the lines joining corresponding points of two homographic ranges meet in a point, the ranges are said to be in **perspective**, and the point of concurrence is called the **centre of perspective**.

24. From the preceding article it is easy to see how to construct the points of two homographic ranges.

Let P, Q, R and P', Q', R' be the points which determine the ranges. Join $P'Q$, PQ', $Q'R$, QR', and so determine the Homographic Axis KK'. (Fig., Art. 19.)

Take any point X on the axis of the first range; join XR' to meet the homographic axis in U, and RU to meet the axis of the second range in X'. Then X' is the point corresponding to X. By this method we can determine as many points as we wish, and so the ranges are known when three points of each are given.

The vanishing points I, J' are obtained as particular cases of the foregoing construction. For I corresponds to the point ∞' at infinity on the second range so that, if we take any two corresponding points Q and Q', then $Q\infty'$ and $Q'I$ meet on the homographic axis.

Hence, if we draw $Q\infty'$ parallel to the second range to meet the homographic axis in V, then $Q'V$ by its intersection with the axis of the first range gives I.

Similarly, draw $Q'W$ parallel to the first range to meet the homographic axis in W, and then QW by its intersection with the axis of the second range gives J'.

Ex. Draw the figure of Art. 19 for two ranges where corresponding points are connected by the equation $x_1y_1 - 2x_1 - 3y_1 - 6 = 0$.

25. *The locus of the intersection of corresponding rays of two homographic pencils is a conic section.*

Let O and O' be the vertices of the two pencils. Take O as the origin, OO' as the axis of x, and let $OO' = c$.

Let the intersection of two corresponding rays OP and $O'P$ be P, and let their equations be

$$y = mx, \quad \text{and} \quad y = m'(x - c).$$

Since the pencils are homographic, we have, as in Art. 17,

$$Amm' + Bm + Cm' + D = 0.$$

Eliminating m and m', the locus of P is the conic

$$Ay^2 + By(x - c) + Cxy + Dx(x - c) = 0,$$

i.e. $$Dx^2 + (B + C)xy + Ay^2 - c(Dx + By) = 0 \quad\ldots\ldots(1),$$

which passes through O and O'.

The tangent at O is $y = -\frac{D}{B}x$. But $-\frac{D}{B}$ is the value of m which corresponds to $m' = 0$, *i.e.* to $O'O$. Hence the tangent at O corresponds to the ray $O'O$, and the tangent at O' similarly corresponds to the ray OO'.

If $D = 0$, the conic breaks up into two straight lines, viz. $Ay + (B + C)x - Bc = 0$, and $y = 0$, the line joining the vertices of the pencil. In this case, when the intersection of corresponding rays is a straight line, the pencils are said to be in **perspective**, and the straight line is called the **axis of perspective**. When this is the case, *i.e.* when $D = 0$, then m is zero when m' is zero. Hence OO' is the ray corresponding to $O'O$, so that the line joining the two vertices is a common ray, *i.e.* a ray corresponding to itself in the two pencils.

Ex. 1. Two homographic ranges are taken on the axes of x and y; three points on the first range at distances 0, 1, 3 from the origin correspond to three points on the second range distant 1, 4, 8 respectively from the origin. Shew that the general relation between the distances of corresponding points from the origin is

$$x_1y_1 - 22x_1 + 6y_1 - 6 = 0,$$

that the equation to the axis of homography is $11x - 3y + 3 = 0$, and the equation to the conic enveloped by lines joining corresponding points is $(11x - 3y + 3)^2 + 126xy = 0$.

[The student would do well to verify this by plotting out a number of points belonging to the system and drawing the corresponding lines.]

Ex. 2. Two homographic pencils of rays have their vertices at the points (1, 3) and (2, 4). The rays given by $x = 1$, $y = 3x$, $x - y + 2 = 0$ of the first pencil correspond to the rays $y = 2x$, $y = 4$, $x + y = 6$ of the second pencil. Shew that the ray $y + 3x = 6$ of the first pencil corresponds to the ray $x = 2$ of the second pencil, and that the equation to the conic given by the intersection of corresponding rays of the pencils is $6x^2 + xy + y^2 - 24x - 6y + 24 = 0$.

[Prove that the relation between m_1 and m_2 is

$$m_1m_2 - 2m_1 + 3m_2 + 6 = 0.]$$

Ex. 3. Consider what the theorem of this article becomes if $AD = BC$.

Ex. 4. Shew that the straight line IJ', joining the vanishing points of two ranges, is parallel to KK', the line joining the points corresponding to the point of intersection of the two ranges.

Coaxal Homographic Ranges.

26. Let the axes of the two ranges in Art. 19 coincide, so that the two ranges of points are on the same straight line, and let the distances, x and x', be measured from the same origin O. A point E, looked upon as belonging to the first range, may now have itself as its corresponding point in the second range. This will be the case when $x = x'$, and then the fundamental relation

$$Axx' + Bx + Cx' + D = 0 \ldots\ldots\ldots\ldots\ldots(1)$$

of Art. 19 becomes

$$Ax^2 + (B + C)x + D = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

This equation gives, in general, two values for x, and hence two positions of E, viz. E and F.

The distance of the middle point of EF from O

$$= \frac{OE + OF}{2} = -\frac{B + C}{2A} = \tfrac{1}{2}(OI + OJ').$$

Hence the middle point of the common, or self-corresponding, points coincides with the middle point of the two points which correspond to the points at infinity, *i.e.* of the two vanishing points. These self-corresponding points, E and F, are often called the double points.

27. If we transfer the origin to the point O_1 midway between the double points, *i.e.* if we write $x - \frac{B + C}{2A}$ and $x' - \frac{B + C}{2A}$ for x and x', then (1) becomes

$$Axx' + \frac{B - C}{2}(x - x') + D - \frac{(B + C)^2}{4A} = 0,$$

i.e.

$$Axx' + B_1(x - x') + D_1 = 0.$$

The distances of the double points E, F from O_1 are now $\pm\lambda$, where $\lambda^2 = -\frac{D_1}{A}$.

Hence, if x_1, x_2 are the distances from O_1 of two corresponding points P_1, P_2, then

$$(EP_1FP_2)=\frac{(x_1+\lambda)(x_2-\lambda)}{(\lambda-x_1)(-\lambda-x_2)}=\frac{x_1x_2+\lambda(x_2-x_1)-\lambda^2}{x_1x_2+\lambda(x_1-x_2)-\lambda^2}$$

$$=\frac{(A\lambda+B_1)(x_2-x_1)}{(A\lambda-B_1)(x_1-x_2)}=\frac{B_1+A\lambda}{B_1-A\lambda}=\frac{B_1+\sqrt{-AD_1}}{B_1-\sqrt{-AD_1}}=\text{const.}$$

Hence *the cross-ratio of the range formed by the double points of two coaxal homographic ranges and any pair of corresponding points is constant.*

Ex. 1. If O_1 be the middle point of IJ', O' the point corresponding to O_1, the common points E, F are given by

$$O_1E^2=O_1F^2=O_1J'\,.\,O_1O'.$$

Ex. 2. Two coaxal ranges have imaginary common points, and O_1 is the real middle point of these imaginary points; K is a point on the line through O_1 perpendicular to the axis of the ranges; if O_1K be properly chosen, shew that each pair of corresponding points of the ranges subtends the same angle at K, and hence the pencil subtended at K by one range can be rotated about K to fit on to the pencil subtended at K by the other range.

$$\left[O_1K=\pm\sqrt{\frac{D_1}{A}}.\right]$$

Involution Ranges.

28. We have seen that two homographic ranges on the same line are given by an equation of the form

$$Axx'+Bx+Cx'+D=0,$$

so that $\quad x'=-\dfrac{Bx+D}{Ax+C}$, and $x=-\dfrac{Cx'+D}{Ax'+B}$.

Take any point P whose distance from the origin is ξ.

If P belongs to the first range of points, the distance of its corresponding point

$$=x'=-\frac{B\xi+D}{A\xi+C}.$$

If P is looked upon as belonging to the second range, the distance of its corresponding point

$$=x=-\frac{C\xi+D}{A\xi+B}.$$

In general these distances are different; but they are the same if

$$\frac{B\xi + D}{A\xi + C} = \frac{C\xi + D}{A\xi + B},$$

i.e. if $\quad A\xi^2 (B - C) + (B^2 - C^2)\,\xi + (B - C)\,D = 0.$

This is true for all values of ξ if $B = C$; hence, if $B = C$, the corresponding point of every point P is the same whether it is looked upon as belonging to the first or the second ranges of points.

Such a range of points is said to be in **involution**.

29. The fundamental relation for points in involution is thus

$$Axx' + B(x + x') + D = 0,$$

i.e.
$$\left(x + \frac{B}{A}\right)\left(x' + \frac{B}{A}\right) = \frac{B^2 - AD}{A^2}.$$

The origin being the point O_1, if we take O so that

$$O_1O = -\frac{B}{A},$$

this relation gives

$$OP \, . \, OP' = \frac{B^2 - AD}{A^2} = \text{constant} = k^2.$$

The point O is called the **centre** of the involution and P, P' are a pair of conjugate points or **mates**.

If k^2 is positive, so that P and P' are on the same side of the centre O, there will be two points, K and K', on opposite sides of O, such that $OK^2 = OK'^2 = k^2$. These points, K and K', are called the double points or foci of the involution.

If k^2 is negative, these double points are imaginary, and corresponding points P, P' are on opposite sides of the centre O.

Since $OK^2 = OK'^2 = OP \, . \, OP'$, therefore, as in Art. 6, $[KPK'P']$ is harmonic.

30. *If a number of points are in involution, the cross-ratio of any four points is equal to that of their conjugates.*

Let the four points be P, Q, R, S and their conjugates P', Q', R', S' so that, if O is the centre of the involution,

$$OP.OP'=OQ.OQ'=OR.OR'=OS.OS'=k^2.$$

Therefore

$$[P'Q'R'S']=\frac{OQ'-OP'}{OR'-OQ'}\cdot\frac{OS'-OR'}{OP'-OS'}=\frac{\dfrac{k^2}{OQ}-\dfrac{k^2}{OP}}{\dfrac{k^2}{OR}-\dfrac{k^2}{OQ}}\cdot\frac{\dfrac{k^2}{OS}-\dfrac{k^2}{OR}}{\dfrac{k^2}{OP}-\dfrac{k^2}{OS}}$$

$$=\frac{OQ-OP}{OR-OQ}\cdot\frac{OS-OR}{OP-OS}=[PQRS].$$

31. *If six points are in involution, the cross-ratio of any four of the six points (provided they do not form two pairs of mates) is equal to the cross-ratio of their four mates.*

Let the six points be P, Q, R, P', Q', R', where

$$OP.OP'=OQ.OQ'=OR.OR'=k^2.$$

Then

$$[PQRP']=\frac{OQ-OP}{OR-OQ}\cdot\frac{OP'-OR}{OP-OP'}=\frac{\dfrac{k^2}{OQ'}-\dfrac{k^2}{OP'}}{\dfrac{k^2}{OR'}-\dfrac{k^2}{OQ'}}\cdot\frac{\dfrac{k^2}{OP}-\dfrac{k^2}{OR'}}{\dfrac{k^2}{OP'}-\dfrac{k^2}{OP}}$$

$$=\frac{OQ'-OP'}{OR'-OQ'}\cdot\frac{OP-OR'}{OP'-OP}=[P'Q'R'P].$$

[This can be shewn to be true of any two sets of homographic points only when $B=C$, and this is the definition in Art. 28 of points in involution.]

32. *In an involution range the mates, I and J', of points at infinity coincide.*

For $Axx'+B(x+x')+D=0$ (Art. 29).

$$\therefore\ x'=-\frac{Bx+D}{Ax+B},\ \text{ and }\ x=-\frac{Bx'+D}{Ax'+B}.$$

Hence when $x=\infty$, $x'=-\frac{B}{A}$, giving J', and when $x'=\infty$, $x=-\frac{B}{A}$, giving I.

Hence I and J' coincide.

33. The four properties given in Arts. 28–32 have been taken by different writers as definitions of points in involution. It is thus seen that the last three follow from the definition of Art. 28.

34. The fundamental relation connecting pairs of corresponding points in involution (Art. 29) contains only two arbitrary constants, viz. $\frac{A}{D}$ and $\frac{B}{D}$. They are thus determined when two relations between them are given.

This is the case when two pairs of corresponding points are given. Thus only two pairs of points are required to determine an involution, whilst three pairs were required to determine two homographic ranges.

The constants are also determined when the two double points are given; for a double point is such that it corresponds to itself (*i.e.* x and x' are the same in this case) and is thus given by $Ax^2+2Bx+D=0$.

If both double points are given, *i.e.* if both roots of this equation are given, we have the two constants.

These constants are also determined when the centre and one double point are given; for then, by Art. 29, $-\frac{B}{A}$ and $\frac{B^2-AD}{A^2}$ are given.

Similarly it may be seen that the involution is determined when a pair of mates and the centre are given, or when a pair of mates and either double point is given.

35. The relation given in Art. 29 for a range of points in involution holds similarly for a pencil of lines in involution, *i.e.* for a pencil which is cut by a transversal in points which are in involution.

For if a pencil of lines $y=px$, $y=p'x$, etc., in involution is cut by any straight line parallel to the axis of y, we have a range of points in involution whose distances from the axis of x are proportional to p, p', etc., so that the relation between any pair of lines belonging to a pencil in involution is

$$App'+B(p+p')+D=0 \qquad \text{..............(1)},$$

where $y=px$ and $y=p'x$ are the equations to a pair of conjugate lines.

The double rays, *i.e.* the rays to the double points of the involution range, are given by

$$Ap^2+2Bp+D=0.$$

In the particular case where the axes of x and y are the double rays of the involution pencil, this equation is satisfied by $p=0$ and $p=\infty$. Hence both A and D are zero, and the fundamental relation (1) reduces to $p+p'=0$.

There is no ray in an involution pencil which can be called the central ray of the involution, and so no ray to correspond to the centre of an involution range.

36. *Involution Ranges are given by Coaxal Circles.*

If we have a system of coaxal circles (as in Part I, Page 168) and if from any point O' on the radical axis we draw a straight line to cut the circles whose centres are $O_1, O_2, O_3, \ldots$ in the points P and P', Q and Q', R and R', ..., we have $OP.OP'=OQ.OQ'=OR.OR'\ldots=k^2$, where k is the length of the tangent from O' to any of the circles of the system.

These pairs of points therefore form an involution range, and the double points are clearly the points where the axis of this range touches two of the circles of the coaxal system.

If instead of any point O' on the radical axis we take the point of intersection, O, of the radical axis and the line of centres as the axis of the range, we have an involution pencil the double points of which are the limiting points of the coaxal system of circles. With the figure of page 168 these double points are real; with that of page 167 these double points are imaginary.

Conversely, if we wish to construct pairs of points belonging to the involution determined by P, P' and Q, Q' as two pairs of mates, through P, P' and Q, Q' draw any two suitable circles which intersect in two points U, V. Then the circle through U, V and any point R of the involution will pass through R', the mate of R, and hence we can determine as many points as we wish. The double points are the points of contact of the axis $PP'QQ'$ with the two circles through U and V which touch it, and the centre of the involution is the point of intersection of UV with this axis.

37. *In every involution pencil to shew that there is one pair of conjugate rays at right angles, and, if there is more than one pair at right angles, then every pair of conjugate rays is at right angles.*

With rectangular axes through the vertex of the pencil, let $y=px$ and $y=p_1x$ be conjugate rays. Then

$$App_1+B(p+p_1)+C=0 \text{..............} (1).$$

If there is a pair at right angles then $pp_1=-1$, and (1) gives $A-C=B\left(p-\frac{1}{p}\right)$, so that the pair at right angles is given by $Bp^2+(C-A)p-B=0$.

If there is more than one pair at right angles this equation must be identically satisfied, *i.e.* $C=A$ and $B=0$. In this case the general relation (1) between the rays becomes

$$pp_1=-1,$$

i.e. each pair of conjugate rays is at right angles.

Such a pencil in which each pair of conjugate rays is at right angles is called an orthogonal involution pencil.

Cor. The double rays of this orthogonal involution are given by $p^2=-1$, *i.e.* by $x^2+y^2=0$, so that the imaginary lines drawn to the circular points at infinity are the double rays of every orthogonal involution pencil, and conversely an involution pencil having the lines to the circular points at infinity as double lines is orthogonal.

38. *To shew that the centre of the involution determined by the points given by* $a_1x^2+2h_1x+b_1=0$ *and* $a_2x^2+2h_2x+b_2=0$ *is the point whose abscissa is* $\frac{1}{2}\frac{a_1b_2-a_2b_1}{h_1a_2-h_2a_1}$, *and that the double points are given by the equation*

$$x^2(h_1a_2-h_2a_1)-x(a_1b_2-a_2b_1)+b_1h_2-b_2h_1=0.$$

Let the relation connecting a pair of conjugate points be

$$Ax_1x_2+B(x_1+x_2)+C=0.$$

Hence $$Ab_1-2Bh_1+Ca_1=0,$$

and $$Ab_2-2Bh_2+Ca_2=0,$$

and the double points are given by $Ax^2+2Bx+C=0$.

Eliminating A, B, C, the double points are given by

$$x^2(h_1a_2 - h_2a_1) - x(a_1b_2 - a_2b_1) + b_1h_2 - b_2h_1 = 0 \ldots (1).$$

Also, the centre of the involution being midway between the double points, its abscissa $= \frac{1}{2}\frac{a_1b_2 - a_2b_1}{h_1a_2 - h_2a_1}$.

Clearly the double points are imaginary when the roots of (1) are imaginary, but the centre is always real.

39. *To find the condition that the points given by*

$$a_1x^2 + 2h_1x + b_1 = 0, \quad a_2x^2 + 2h_2x + b_2 = 0$$

and $a_3x^2 + 2h_3x + b_3 = 0$ *shall form an involution range.*

Let the relation connecting pairs of points in the involution range be $Ax_1x_1' + B(x_1 + x_1') + C = 0$.

Since each of the above pairs of points satisfies this relation,

$$\therefore Ab_1 - 2Bh_1 + Ca_1 = 0,$$

$$Ab_2 - 2Bh_2 + Ca_2 = 0,$$

and

$$Ab_3 - 2Bh_3 + Ca_3 = 0.$$

Eliminating A, B, and C, we have as the required condition

$$\begin{vmatrix} a_1, & h_1, & b_1 \\ a_2, & h_2, & b_2 \\ a_3, & h_3, & b_3 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Cor. 1. It follows that all pairs of points belonging to the involution determined by the first two pairs of points is given by

$$a_1x^2 + 2h_1x + b_1 + \lambda(a_2x^2 + 2h_2x + b_2) = 0,$$

where λ is a variable parameter.

For clearly (1) is satisfied if

$$a_3 = a_1 + \lambda a_2, \; h_3 = h_1 + \lambda h_2, \text{ and } b_3 = b_1 + \lambda b_2.$$

Cor. 2. Similarly a pencil of rays given by the equations

$$a_1x^2 + 2h_1xy + b_1y^2 = 0, \quad a_2x^2 + 2h_2xy + b_2y^2 = 0$$

and

$$a_3x^2 + 2h_3xy + b_3y^2 = 0$$

is in involution if the condition (1) holds.

40. *A system of conics passing through four fixed points is cut by any transversal in a system of points which are in involution.*

Take the transversal as the axis of x, and let two conics through the four given points be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and $$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0.$$

Then $S + \lambda S' = 0$ denotes any other conic of the system, and is met by the transversal, $y = 0$, in points given by

$$(ax^2 + 2gx + c) + \lambda (a'x^2 + 2g'x + c') = 0.$$

By Art. 39, Cor. 1, this denotes a range of pairs of points which are in involution.

Taking as particular cases the three pairs of straight lines which can be drawn through the four points, we see that *any transversal drawn across the quadrangle formed by the four points cuts the three pairs of opposite sides in six points which are in involution.*

41. *The locus of the intersection of two tangents to a given conic, drawn from conjugate points of an involution, is a conic which passes through the double points of the involution and through the points of contact of tangents drawn to the given conic from the double points of the involution. If the axis of the involution touches the given conic, then the locus is a straight line.*

Take the axis of the involution as the axis of x, and the centre of the involution as the origin, so that corresponding points of the involution are given by $x_1 x_2 = k^2$, and the double points are $(\pm k, 0)$.

Any conic is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. The tangents to it drawn from (x', y') meet the axis of x where

$$(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c)(ax^2 + 2gx + c)$$
$$= [x(ax' + hy' + g) + (gx' + fy' + c)]^2.$$

This is a quadratic for x, and the product of its roots is given to be k^2.

$$\therefore \; k^2 = \frac{(ca - g^2)x'^2 - 2(fg - ch)x'y' + (bc - f^2)y'^2}{(ab - h^2)y'^2 - 2(gh - uf)y' + ca - g^2}.$$

Hence the locus of $(x'y')$ is the conic

$$(ca - g^2)x^2 - 2(fg - ch)xy + (bc - f^2)y^2$$
$$- k^2[(ab - h^2)y^2 - 2(gh - af)y + ca - g^2] = 0.$$

This clearly passes through the double points $(\pm k, 0)$. Also clearly the point of contact of a tangent from a double point to the given conic must lie on the locus; for from two (coinciding) points we have drawn tangents and their point of contact is their point of intersection.

If the axis of x touches the conic, then $g^2=ac$, and the locus reduces to the straight line

$$-2(fg-ch)x+(bc-f^2)y-k^2[(ab-h^2)y-2(gh-af)]=0,$$

which passes through the point of the involution which is conjugate to the point of contact of the conic with the axis of the involution.

EXAMPLES.

1. If one pair of conjugate points of an involution is at distances α and α' from a fixed origin, and a second pair is at distances β and β', shew that the distance of the centre of the involution is $\dfrac{\alpha\alpha'-\beta\beta'}{\alpha+\alpha'-\beta-\beta'}$, and that the constant of the involution is $\dfrac{(\alpha-\beta)(\alpha-\beta')(\alpha'-\beta)(\alpha'-\beta')}{(\alpha+\alpha'-\beta-\beta')^2}$.

[We have $(\alpha-x)(\alpha'-x)=(\beta-x)(\beta'-x)=k^2$.]

2. If A and A' are a given pair of mates in an involution, and P and P' any other pair of mates, shew that the ratio of $AP.AP'$ to $A'P.A'P'$ is constant, and equal to the ratio of the distances of A and A' from the centre of the involution.

3. Shew that the harmonic conjugates of a variable point with regard to four given pairs of an involution of points have a constant cross-ratio.

4. Shew that three pairs of points which are harmonic conjugates of a given pair of points form a system in involution.

5. If A, A' and B, B' are two pairs of conjugate points of an involution, and C, C' are the common harmonic conjugates of A, B and A', B', shew that they are a pair of conjugate points of the involution.

6. If A and A_1, B and B_1, O and U are point pairs in involution, as also A' and A_1, B' and B_1, O and U, then prove that A and B', A' and B, O and U are point pairs in involution.

7. The double points of the three involutions determined from any six collinear points by the pairs BC' and $B'C$, CA' and $C'A$, and AB' and $A'B$ are themselves in involution.

8. Given four collinear points, shew that they determine three involutions. Shew also that the double points of any one of these involutions are harmonic conjugates with respect to the double points of any other.

9. If $ABA'B'$ is a harmonic range, and L, M are the harmonic conjugates of a point P with respect to A and A', B and B', prove that L, M are a pair of conjugate points of the involution determined by A and A', B and B'.

10. The points P, Q, R in which a straight line meets the sides of a triangle ABC, and the points P', Q', R' in which it meets the lines

joining any point O to the corresponding angular points of the triangle, form a range in involution, and hence

$$PQ' . QR' . RP' = -P'Q . Q'R . R'P.$$

[Use Art. 40.]

11. Shew that pairs of conjugate points with respect to a conic (*i.e.* points such that the polar of each passes through the other) lying on any given line form a range in involution, whose double points are the intersection of the straight line and the conic.

[Take the given straight line as the axis of x, and the conic as given by the general equation. Then the polar of $(x_1, 0)$ passes through $(x_2, 0)$ if $ax_1x_2 + g(x_1 + x_2) + c = 0$. Hence, etc.]

12. Shew that pairs of conjugate lines with respect to a conic (*i.e.* lines such that the pole of each lies on the other) drawn through any given point form a pencil in involution, whose double rays are the tangents to the conic from the given point.

[Take the given point as origin, and $y = px$, $y = p'x$ as a pair of conjugate lines. By Part I, Art. 375, the pole (x', y') of the line $y - px = 0$ is given by $ax' + hy' + g = -p(hx' + by' + f)$ and $gx' + fy' + c = 0$. If this pole lies on $y - p'x = 0$, we have also $y' - p'x' = 0$. Hence, on eliminating (x', y'), we have $(bc - f^2)pp' - (fg - ch)(p + p') + ca - g^2 = 0$, so that the lines always form a pencil in involution. Clearly the double lines are the tangents from the point.]

13. Conjugate diameters of a conic form an involution pencil whose double lines are the asymptotes. [Deduce from the preceding, or prove independently from Part I, Art. 376.]

14. Shew that the envelope of a chord of a given conic, whose ends lie on corresponding rays of a pencil in involution, is a conic touching the double rays of the involution, and also touching the tangents drawn to the given conic at the points where the double rays meet it.

If the given conic passes through the vertex of the involution pencil, then the envelope reduces to a point.

[Take the vertex of the pencil as origin, and the axes as the double rays of the pencil. (Art. 35.)]

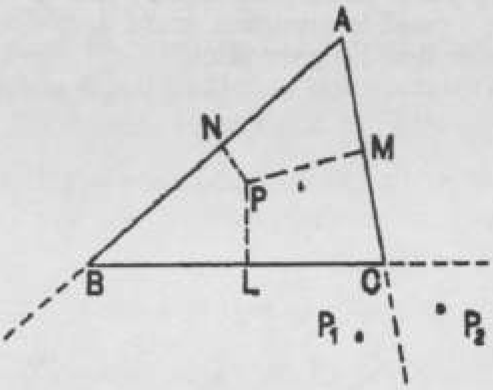
15. The locus of the intersection of tangents to a conic drawn from corresponding points of two homographic ranges on the same straight line is a curve of the fourth degree, which reduces to a conic having double contact with the given conic if the axis of the ranges touches the given conic.

16. The envelope of a straight line, such that its intersections with a given conic are harmonic conjugates with respect to its intersections with a pair of given lines, is a conic which touches the pair of lines. [Take the given lines as the axes of x and y.]

CHAPTER II

TRILINEAR AND AREAL COORDINATES. THE STRAIGHT LINE

42. In the following chapter we shall introduce a new system of coordinates, which has been foreshadowed in the notation of Part I, Arts. 409, 410.



Let ABC be a fixed triangle known as the "triangle of reference."

From any point P in its plane draw straight lines PL, PM, and PN perpendicular respectively to BC, CA, and AB. These three distances PL, PM, and PN are called α, β, and γ, and are known as the trilinear coordinates of P.

Any such coordinate, α, is positive if P and the vertex A of the triangle are on the same side of BC, and is negative if P and A are on opposite sides of BC. So for the other coordinates. Thus in the figure P has all its coordinates positive; for P_1, its α is negative and its β and γ are positive; whilst, for P_2, its α and β are both negative, whilst its γ is positive. So for any other position of the point P.

43. *Whatever be the position of the point P, if a, b and c are the lengths of the sides BC, CA and AB, and if Δ be the area of the triangle, then $a\alpha + b\beta + c\gamma = 2\Delta$.*

If P be within the triangle, then

$$a\alpha + b\beta + c\gamma = PL \times BC + PM \times CA + PN \times AB$$
$$= 2\triangle PBC + 2\triangle PCA + 2\triangle PAB = 2\triangle ABC = 2\Delta.$$

If P be outside the triangle ABC but within the area formed by AB, AC as at P_1, then α is negative, so that

$$a\alpha = -2\triangle P_1BC,$$

and hence for this point

$$a\alpha + b\beta + c\gamma = -2\triangle P_1BC + 2\triangle P_1CA + 2\triangle P_1AB = 2\Delta,$$

as before. If P have a position such as P_2, where both α and β are negative, then

$$a\alpha + b\beta + c\gamma = -2\triangle P_2BC - 2\triangle P_2CA + 2\triangle P_2AB = 2\Delta.$$

Similarly any other point can be considered. Hence, if P be any point within a finite distance of the triangle of reference, we have this important relation always true.

44. By its use we can always make any equation in trilinear coordinates take a homogeneous form. For example, suppose we had an equation

$$l\alpha^2 + m\gamma^2 + 2n\alpha\beta + q\alpha + r\beta + s\gamma + K = 0 \text{ ......(1)},$$

where l, m, n, q, r, s, K are all constants. Since $\dfrac{a\alpha + b\beta + c\gamma}{2\Delta}$ is always unity, we can multiply any terms we please in equation (1) by this quantity, or any power of it, without altering the equation. Multiply therefore the terms of the first degree in (1) by this quantity, and the constant terms by its square, and we have

$$(l\alpha^2 + m\gamma^2 + 2n\alpha\beta) + (q\alpha + r\beta + s\gamma) \times \frac{a\alpha + b\beta + c\gamma}{2\Delta}$$
$$+ K\left(\frac{a\alpha + b\beta + c\gamma}{2\Delta}\right)^2 = 0,$$

i.e.

$$4\Delta^2 (l\alpha^2 + m\gamma^2 + 2n\alpha\beta) + 2\Delta (q\alpha + r\beta + s\gamma)(a\alpha + b\beta + c\gamma)$$
$$+ K(a\alpha + b\beta + c\gamma)^2 = 0,$$

which is a homogeneous equation of the second degree in α, β, and γ.

In trilinear equations we almost invariably confine ourselves to homogeneous equations, or to equations which we have made homogeneous by the method of this article.

45. Since our equations are homogeneous, if they are satisfied by the coordinates of any point they are clearly satisfied by any (the same) multiple of these coordinates. Hence we can speak of any quantities, which are proportional to the actual distances of a point from the sides of the triangle, as the coordinates of the point. Thus it is easy to shew that the actual distances of the ortho-centre from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$. We speak of this point as $(\cos B\cos C, \cos C\cos A, \cos A\cos B)$ or again as $(\sec A, \sec B, \sec C)$.

If the coordinates of a point are given as proportional to α', β', γ', we can easily get the corresponding actual numerical coordinates, α_1, β_1, γ_1. For

$$\frac{\alpha_1}{\alpha'} = \frac{\beta_1}{\beta'} = \frac{\gamma_1}{\gamma'} = \frac{a\alpha_1 + b\beta_1 + c\gamma_1}{a\alpha' + b\beta' + c\gamma'} = \frac{2\Delta}{a\alpha' + b\beta' + c\gamma'},$$

so that the actual coordinates are $\dfrac{\alpha'}{a\alpha' + b\beta' + c\gamma'} \times 2\Delta$, and two similar expressions.

Ex. Verify that the coordinates of the following points of the triangle of reference may be taken as the values stated:

In-centre; $(1, 1, 1)$;
Ex-centre opposite A; $(-1, 1, 1)$;
Circum-centre; $(\cos A, \cos B, \cos C)$;
Centroid; $(\sin B\sin C, \sin C\sin A, \sin A\sin B)$;
Ortho-centre; $(\cos B\cos C, \cos C\cos A, \cos A\cos B)$;
Nine-point centre; $[\cos(B-C), \cos(C-A), \cos(A-B)]$.

46. *Relation between Trilinear and Cartesian Coordinates.*

Taking the origin O of rectangular Cartesian Coordinates within the triangle ABC, let the equations to the sides BC, CA, AB of the triangle of reference be

$$x\cos\omega_1 + y\sin\omega_1 - p_1 = 0,\quad x\cos\omega_2 + y\sin\omega_2 - p_2 = 0,$$

and $$x\cos\omega_3 + y\sin\omega_3 - p_3 = 0.$$

Taking the case when $\omega_2 < \omega_3 < \omega_1$, we easily see that

$$\omega_3 - \omega_2 = \pi - A,\quad \omega_1 - \omega_3 = \pi - B,$$

and $$\omega_1 - \omega_2 = 2\pi - (\pi - C) = \pi + C,$$

and hence that

$$\cos(\omega_2 - \omega_3) = -\cos A,\quad \cos(\omega_3 - \omega_1) = -\cos B,$$

and $$\cos(\omega_1 - \omega_2) = -\cos C.$$

The perpendicular α upon BC has the same sign as the perpendicular from the origin O, since the latter is within the triangle. Hence we have

$$\alpha = p_1 - x\cos\omega_1 - y\sin\omega_1,$$

and so also $$\beta = p_2 - x\cos\omega_2 - y\sin\omega_2,$$

and $$\gamma = p_3 - x\cos\omega_3 - y\sin\omega_3.$$

47. By these relations any equation in Trilinear coordinates may be transferred to one in Rectangular Cartesians.

It follows that *Any trilinear equation of the first degree represents a straight line and any equation of the second degree represents a conic section.*

For the most general equation of the first degree is

$$l\alpha + m\beta + n\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where l, m, n are constants. On making the substitutions of the previous article, this equation becomes

$$x(l\cos\omega_1 + m\cos\omega_2 + n\cos\omega_3) \\ + y(l\sin\omega_1 + m\sin\omega_2 + n\sin\omega_3) = lp_1 + mp_2 + np_3,$$

which always represents a straight line.

Also, since (1) has two independent constants $\frac{l}{n}$ and $\frac{m}{n}$, they can be chosen so that the equation is satisfied by the coordinates of any two points, and hence the equation will represent any straight line.

Again, the most general equation of the second degree is

$$L\alpha^2 + M\beta^2 + N\gamma^2 + 2P\beta\gamma + 2Q\gamma\alpha + 2R\alpha\beta = 0 \ldots(2).$$

On making the substitutions of the previous article, we obtain the general equation of the second degree in Cartesian coordinates; hence (2) always represents a conic section.

Also, since (2) has five independent constants, they can be chosen so that the conic passes through any five points, in general, and hence (2) will represent any conic section.

48. *To find the distance between any two points whose trilinear coordinates are given.*

Let P_1 be the point $(\alpha_1, \beta_1, \gamma_1)$ and P_2 the point $(\alpha_2, \beta_2, \gamma_2)$.

Through P_1 draw P_1L and P_1M parallel to AB and BC to meet BC and BA in L and M, and take BC and BA as oblique axes in Cartesian coordinates, so that

$$BL = x_1, \text{ and } LP_1 = y_1.$$

Then

$$\alpha_1 = P_1L \sin P_1LC = y_1 \sin B, \text{ and } \gamma_1 = P_1M \sin P_1MA = x_1 \sin B.$$

Similarly $\alpha_2 = y_2 \sin B$, and $\gamma_2 = x_2 \sin B$,

where (x_2, y_2) are the coordinates of P_2.

By Part I, Art. 20, we have

$$P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos B.$$

$$\therefore P_1P_2^2 . \sin^2 B = (\alpha_1 - \alpha_2)^2 + (\gamma_1 - \gamma_2)^2 + 2(\alpha_1 - \alpha_2)(\gamma_1 - \gamma_2)\cos B \quad \text{......(1)}.$$

We can express this in two forms, both symmetrical in

$$\alpha_1 - \alpha_2,\ \beta_1 - \beta_2, \text{ and } \gamma_1 - \gamma_2.$$

For, by Art. 43,

$$a\alpha_1 + b\beta_1 + c\gamma_1 = 2\Delta = a\alpha_2 + b\beta_2 + c\gamma_2.$$

$$\therefore a(\alpha_1 - \alpha_2) + c(\gamma_1 - \gamma_2) = -b(\beta_1 - \beta_2) \quad \text{......(2)}.$$

Hence, by squaring, we have

$$2ac(\alpha_1 - \alpha_2)(\gamma_1 - \gamma_2) = b^2(\beta_1 - \beta_2)^2 - a^2(\alpha_1 - \alpha_2)^2 - c^2(\gamma_1 - \gamma_2)^2.$$

Hence (1) gives

$$\begin{aligned} P_1P_2^2 . ac\sin^2 B &= ac(\alpha_1 - \alpha_2)^2 + ac(\gamma_1 - \gamma_2)^2 + b^2\cos B(\beta_1 - \beta_2)^2 \\ &\quad - a^2\cos B(\alpha_1 - \alpha_2)^2 - c^2\cos B(\gamma_1 - \gamma_2)^2 \\ &= ab\cos A(\alpha_1 - \alpha_2)^2 + b^2\cos B(\beta_1 - \beta_2)^2 + bc\cos C(\gamma_1 - \gamma_2)^2. \end{aligned}$$

$$\therefore P_1P_2^2 = \frac{\sin A\cos A(\alpha_1 - \alpha_2)^2 + \sin B\cos B(\beta_1 - \beta_2)^2 + \sin C\cos C(\gamma_1 - \gamma_2)^2}{\sin A\sin B\sin C} \quad \text{......(3)}.$$

Again, on multiplying (2) successively by $(\alpha_1 - \alpha_2)$ and by $(\gamma_1 - \gamma_2)$, we have

$$a(\alpha_1 - \alpha_2)^2 = -b(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) - c(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2),$$

and $c(\gamma_1 - \gamma_2)^2 = -b(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) - a(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2).$

Substituting in (1), we have

$$\begin{aligned}P_1P_2^2 \,.\, ac\sin^2 B &= -bc(\alpha_1-\alpha_2)(\beta_1-\beta_2)-c^2(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)\\ &\quad -ab(\beta_1-\beta_2)(\gamma_1-\gamma_2)-a^2(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)\\ &\quad +2(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)\,.\,ac\cos B\\ &= -bc(\alpha_1-\alpha_2)(\beta_1-\beta_2)-ab(\beta_1-\beta_2)(\gamma_1-\gamma_2)\\ &\quad -b^2(\gamma_1-\gamma_2)(\alpha_1-\alpha_2).\end{aligned}$$

$$\therefore\ P_1P_2^2 = -\frac{\left\{\begin{aligned}&(\beta_1-\beta_2)(\gamma_1-\gamma_2)\sin A+(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)\sin B\\ &\qquad\qquad+(\alpha_1-\alpha_2)(\beta_1-\beta_2)\sin C\end{aligned}\right\}}{\sin A\sin B\sin C} \qquad \text{......(4).}$$

Either (3) or (4) gives the required distance.

49. By equation (1) of the last article, the square of the distance of any point $(\alpha_1, \beta_1, \gamma_1)$ from the point B of the triangle of reference is $\dfrac{\alpha_1^2+\gamma_1^2+2\alpha_1\gamma_1\cos B}{\sin^2 B}$. Hence the equation to the circle, whose centre is B and whose radius is r, is

$$\begin{aligned}\alpha^2+\gamma^2+2\alpha\gamma\cos B = r^2\sin^2 B &= \frac{r^2\sin^2 B}{4\Delta^2}(a\alpha+b\beta+c\gamma)^2\\ &= \frac{r^2}{a^2c^2}(a\alpha+b\beta+c\gamma)^2.\end{aligned}$$

Again, the equation to the parabola, whose focus is B and whose directrix is the opposite side CA, is

$$\frac{\alpha^2+\gamma^2+2\alpha\gamma\cos B}{\sin^2 B} = \beta^2.$$

50. *Area of the triangle whose angular points are* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ *and* $(\alpha_3, \beta_3, \gamma_3)$.

Referred to the sides BC, BA of the triangle of reference as axes, let the coordinates of these points be (x_1, y_1), (x_2, y_2), and (x_3, y_3) so that, as in Art. 48, $\alpha_1 = y_1\sin B$ and $\gamma_1 = x_1\sin B$, and so for the others.

If Δ_1 be the required area, then, by Part I, Art. 26,

$$\frac{2\Delta_1}{\sin B} = \begin{vmatrix} x_1, & y_1, & 1\\ x_2, & y_2, & 1\\ x_3, & y_3, & 1\end{vmatrix} = \frac{1}{2\Delta\sin^2 B}\begin{vmatrix}\gamma_1, & \alpha_1, & 2\Delta\\ \gamma_2, & \alpha_2, & 2\Delta\\ \gamma_3, & \alpha_3, & 2\Delta\end{vmatrix}.$$

Multiply the first column by c, the second by a, and subtract their sum from the third column, and we have

$$\Delta_1 = \frac{1}{4\Delta \sin B}\begin{vmatrix} \gamma_1, & \alpha_1, & b\beta_1 \\ \gamma_2, & \alpha_2, & b\beta_2 \\ \gamma_3, & \alpha_3, & b\beta_3 \end{vmatrix} = \frac{b}{4\Delta \sin B}\begin{vmatrix} \gamma_1, & \alpha_1, & \beta_1 \\ \gamma_2, & \alpha_2, & \beta_2 \\ \gamma_3, & \alpha_3, & \beta_3 \end{vmatrix}$$

$$= \frac{abc}{8\Delta^2}\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix},\text{ giving the required area.}$$

51. *Equation to the straight line joining the two points* $(\alpha_1, \beta_1, \gamma_1)$ *and* $(\alpha_2, \beta_2, \gamma_2)$.

Let the required equation be

$$l\alpha + m\beta + n\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots(1).$$

Since it passes through the given points,

$$\therefore\ l\alpha_1 + m\beta_1 + n\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

and

$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(3).$$

Solving (2) and (3), we have

$$\frac{l}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{m}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{n}{\alpha_1\beta_2 - \alpha_2\beta_1},$$

and the required equation is

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0.$$

Or, on eliminating l, m, n from (1), (2) and (3), the equation may be written in the form

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \end{vmatrix} = 0.$$

52. *Condition that three given points* $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ *and* $(\alpha_3, \beta_3, \gamma_3)$ *may lie on a straight line.*

The straight line $l\alpha + m\beta + n\gamma = 0$ passes through the three points if

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0,$$

$$l\alpha_2 + m\beta_2 + n\gamma_2 = 0,$$

and

$$l\alpha_3 + m\beta_3 + n\gamma_3 = 0.$$

Eliminating l, m, n from these equations, we have

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix} = 0,$$

which is the required condition.

53. *Coordinates of the point of intersection of the two straight lines whose equations are* $l\alpha + m\beta + n\gamma = 0$ *and* $l'\alpha + m'\beta + n'\gamma = 0$.

Both equations are satisfied at their point of intersection $(\alpha', \beta', \gamma')$. Hence, solving them, we have

$$\frac{\alpha'}{mn' - m'n} = \frac{\beta'}{nl' - n'l} = \frac{\gamma'}{lm' - l'm}$$

$$= \frac{2\Delta}{a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm)} \quad \ldots(1)$$

as in Art. 43.

These coordinates are infinite, *i.e.* the two straight lines are parallel, if the denominator of the right-hand member of (1) is zero, *i.e.* if

$$a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm) = 0.$$

54. *To find the condition that two straight lines may be perpendicular.*

Let the equations of the lines be

$$l\alpha + m\beta + n\gamma = 0 \text{ and } l'\alpha + m'\beta + n'\gamma = 0.$$

On making the substitutions of Art. 46, the Cartesian equations of the straight lines are

$$x(l\cos\omega_1 + m\cos\omega_2 + n\cos\omega_3)$$
$$+ y(l\sin\omega_1 + m\sin\omega_2 + n\sin\omega_3) - (lp_1 + mp_2 + np_3) = 0,$$

and

$$x(l'\cos\omega_1 + m'\cos\omega_2 + n'\cos\omega_3)$$
$$+ y(l'\sin\omega_1 + m'\sin\omega_2 + n'\sin\omega_3) - (l'p_1 + m'p_2 + n'p_3) = 0.$$

By Part I, Art. 69, these are perpendicular if

$$(l\cos\omega_1 + m\cos\omega_2 + n\cos\omega_3)(l'\cos\omega_1 + m'\cos\omega_2 + n'\cos\omega_3)$$
$$+ (l\sin\omega_1 + m\sin\omega_2 + n\sin\omega_3)(l'\sin\omega_1 + m'\sin\omega_2 + n'\sin\omega_3) = 0,$$

i.e. if

$$ll' + mm' + nn' + (mn' + m'n)\cos(\omega_2 - \omega_3) + \\ (nl' + n'l)\cos(\omega_3 - \omega_1) + (lm' + l'm)\cos(\omega_1 - \omega_2) = 0,$$

i.e., by Art. 46, if

$$\mathbf{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B} \\ \mathbf{- (lm' + l'm)\cos C = 0.}$$

Cor. The equation

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0,$$

when it represents straight lines, is equivalent to

$$(l\alpha + m\beta + n\gamma)(l'\alpha + m'\beta + n'\gamma) = 0,$$

so that

$$\frac{ll'}{a} = \frac{mm'}{b} = \frac{nn'}{c} = \frac{mn' + m'n}{2f} = \frac{nl' + n'l}{2g} = \frac{lm' + l'm}{2h},$$

and hence the straight lines are at right angles if

$$a + b + c - 2f\cos A - 2g\cos B - 2h\cos C = 0.$$

55. The angle θ between the two lines of the previous article may be obtained. For by Part I, Art. 66, we have

$$\tan\theta = \frac{A_2B_1 - A_1B_2}{A_1A_2 + B_1B_2}.$$

In the case of the two lines of the previous articles the numerator

$$= (l'\cos\omega_1 + m'\cos\omega_2 + n'\cos\omega_3)(l\sin\omega_1 + m\sin\omega_2 + n\sin\omega_3) \\ - (l\cos\omega_1 + m\cos\omega_2 + n\cos\omega_3)(l'\sin\omega_1 + m'\sin\omega_2 + n'\sin\omega_3)$$

$$= (m'n - mn')\sin(\omega_3 - \omega_2) + (n'l - nl')\sin(\omega_1 - \omega_3) \\ + (l'm - lm')\sin(\omega_2 - \omega_1)$$

$$= (m'n - mn')\sin A + (n'l - nl')\sin B + (l'm - lm')\sin C,$$

since, by Art. 46,

$$\omega_3 - \omega_2 = \pi - A,\ \omega_1 - \omega_3 = \pi - B, \text{ and } \omega_1 - \omega_2 = \pi + C.$$

Also the denominator is as in the last article. Hence

$$\tan\theta = \frac{(m'n - mn')\sin A + (n'l - nl')\sin B + (l'm - lm')\sin C}{\left\{\begin{array}{r} ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B \\ - (lm' + l'm)\cos C \end{array}\right\}}.$$

The two lines are parallel if $\theta = 0$, so that the numerator of the right-hand side is zero.

Thus the two lines are parallel if

$$a(mn' - m'n) + b(nl' - n'l) + c(lm' - l'm) = 0.$$

56. *Intercepts of a straight line on the sides of the triangle of reference.*

Let the straight line, whose equation is

$$l\alpha + m\beta + n\gamma = 0 \quad \text{..................(1)},$$

meet the sides BC, CA, AB in the points P, Q, R. The coordinates of P are O, $CP \sin C$, and $BP \sin B$ so that they are proportional to O, $c(a-x)$, and bx, where $x = BP$.

Hence (1) gives

$$mc(a-x) + nbx = 0,$$

$$\therefore x = BP = \frac{mca}{mc - nb} = \frac{\dfrac{ma}{b}}{\dfrac{m}{b} - \dfrac{n}{c}}.$$

So $$CQ = \frac{\dfrac{nb}{c}}{\dfrac{n}{c} - \dfrac{l}{a}} \quad \text{and} \quad AR = \frac{\dfrac{lc}{a}}{\dfrac{l}{a} - \dfrac{m}{b}}.$$

If $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$, so that the equation of the given line becomes $a\alpha + b\beta + c\gamma = 0$, all three of the above quantities are infinite, *i.e.* the given straight line meets each of the sides of the triangle of reference at an infinite distance from A, B and C respectively. This straight line is known as the **Line at Infinity.**

In Part I, Art. 386, we have shewn that the equation to the line at infinity in Cartesian coordinates is

$$O \cdot x + O \cdot y + C = 0$$

or, more shortly, $C = 0$.

In Art. 43 we have shewn that for all points at a finite distance from the triangle of reference we always have the relation

$$a\alpha + b\beta + c\gamma = \text{a const.}$$

In the case of trilinears the equation to the line at infinity thus reduces to $C=0$ also.

The relation $a\alpha+b\beta+c\gamma=0$ is satisfied by quantities which are proportional to the coordinates of any infinitely distant point.

57. *Condition that two straight lines may be parallel.*

Let the two straight lines be

$$l\alpha+m\beta+n\gamma=0 \quad\text{..................(1),}$$

and

$$l'\alpha+m'\beta+n'\gamma=0 \quad\text{..................(2).}$$

In Part I, Art. 386, we have seen that parallel straight lines meet on the line at infinity. Hence these two straight lines meet on

$$a\alpha+b\beta+c\gamma=0 \quad\text{..................(3).}$$

Eliminating α, β, γ from (1), (2), (3), we have

$$\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ a, & b, & c \end{vmatrix}=0$$

as the condition of parallelism. This has been obtained before in Arts. 53 and 55.

Cor. The equation of the straight line at infinity, $a\alpha+b\beta+c\gamma=0$, satisfies the conditions, of this article and of Art. 54, for being both parallel to and perpendicular to any straight line $l\alpha+m\beta+n\gamma=0$. The explanation is that the direction of the straight line at infinity is quite indefinite.

58. *Straight lines which pass through a given point* $(\alpha', \beta', \gamma')$ *and are respectively* (1) *parallel to, and* (2) *perpendicular to, the straight line* $l\alpha+m\beta+n\gamma=0$.

(1) Any straight line which is parallel to the given straight line passes through its intersection with the line at infinity and hence, as in Part I, Art. 82, has as its equation

$$l\alpha+m\beta+n\gamma+\lambda(a\alpha+b\beta+c\gamma)=0.$$

If, in addition, it passes through the given point, then

$$l\alpha'+m\beta'+n\gamma'+\lambda(a\alpha'+b\beta'+c\gamma')=0.$$

Hence, on elimination of λ, the required equation is

$$\frac{l\alpha + m\beta + n\gamma}{l\alpha' + m\beta' + n\gamma'} = \frac{a\alpha + b\beta + c\gamma}{a\alpha' + b\beta' + c\gamma'}.$$

(2) Let the required line be

$$L\alpha + M\beta + N\gamma = 0 \ldots\ldots\ldots\ldots\ldots(1).$$

Since it passes through the given point,

$$\therefore\ L\alpha' + M\beta' + N\gamma' = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

Since it is perpendicular to the given line, we have

$$L(l - m\cos C - n\cos B) + M(m - n\cos A - l\cos C) \\ + N(n - l\cos B - m\cos A) = 0 \ldots\ldots(3).$$

Solving for L, M, N from (2) and (3) and substituting in (1); or, what is the same thing, eliminating L, M, N from (1), (2) and (3), we have as the equation to the required perpendicular

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ l - m\cos C - n\cos B, & m - n\cos A - l\cos C, & n - l\cos B - m\cos A \end{vmatrix} = 0.$$

59. *Perpendicular distance of a given point $(\alpha', \beta', \gamma')$ from a given straight line $l\alpha + m\beta + n\gamma = 0$.*

When transformed to Cartesian Coordinates, as in Art. 46, the equation of the line becomes

$$x(l\cos\omega_1 + m\cos\omega_2 + n\cos\omega_3) + y(l\sin\omega_1 + m\sin\omega_2 + n\sin\omega_3) \\ - (lp_1 + mp_2 + np_3) = 0.$$

The perpendicular distance of any point (x', y') from this line is obtained (Part I, Art. 75) by substituting x', y' for x, y in this equation and dividing the result by the square root of the sum of the squares of the coefficients of x and y in the equation.

The numerator when rewritten

$$= l(x'\cos\omega_1 + y'\sin\omega_1 - p_1) + m(x'\cos\omega_2 + y'\sin\omega_2 - p_2) \\ + n(x'\cos\omega_3 + y'\sin\omega_3 - p_3)$$

$$= l\alpha' + m\beta' + n\gamma'.$$

The quantity under the square root of the denominator

$$= (l\cos\omega_1 + m\cos\omega_2 + n\cos\omega_3)^2$$
$$+ (l\sin\omega_1 + m\sin\omega_2 + n\sin\omega_3)^2$$
$$= l^2 + m^2 + n^2 + 2mn\cos(\omega_2 - \omega_3)$$
$$+ 2nl\cos(\omega_3 - \omega_1) + 2lm\cos(\omega_1 - \omega_2)$$
$$= l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C$$

(by Art. 46).

Hence the required perpendicular

$$= \frac{\mathbf{l\alpha' + m\beta' + n\gamma'}}{\sqrt{\mathbf{l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C}}}.$$

60. Ex. 1. *Shew that for any triangle ABC the following sets of lines are concurrent:*

(1) *the bisectors of the angles;*

(2) *the lines joining the vertices to the middle points of the opposite sides;*

(3) *the perpendiculars from the angular points upon the opposite sides.*

Let D be the middle point of the side BC, and D_1 and L the points on BC where the bisector of the angle A and the perpendicular from A upon BC meet BC.

If a straight line pass through A its equation must be of the form $\frac{\beta}{\gamma}$ = constant. [For if the general equation $l\alpha + m\beta + n\gamma = 0$ is satisfied by the coordinates of A, where β and γ are both zero, we must have $l = 0$, and hence the equation becomes $\frac{\beta}{\gamma} = -\frac{n}{m} =$ const.]

(1) The equation to AD_1 is

$$\frac{\beta}{\gamma} = \text{const.} = \text{the value of } \frac{\beta}{\gamma} \text{ for any point on } AD_1$$
$$= \text{the value for } D_1 = \frac{D_1A\sin\frac{A}{2}}{D_1A\sin\frac{A}{2}} = 1.$$

The equation to AD_1 is thus $\beta - \gamma = 0$, and those for the corresponding lines BE_1, CF_1 are $\gamma - \alpha = 0$ and $\alpha - \beta = 0$. These all meet at the point $(1, 1, 1)$.

(2) The equation to AD is

$$\frac{\beta}{\gamma} = \text{const.} = \text{the value of } \frac{\beta}{\gamma} \text{ at the point } D = \frac{CD\sin C}{BD\sin B} = \frac{c}{b}.$$

Hence the equation to AD is $b\beta - c\gamma = 0$, and similarly those for the corresponding lines BE, CF are $c\gamma - a\alpha = 0$ and $a\alpha - b\beta = 0$. These all meet where $a\alpha = b\beta = c\gamma$.

(3) The equation to AL is

$$\frac{\beta}{\gamma} = \text{const.} = \frac{CL \sin C}{BL \sin B} = \frac{b \cos C \sin C}{c \cos B \sin B} = \frac{\cos C}{\cos B}.$$

Hence the equation to AL and the two corresponding lines are

$$\beta \cos B - \gamma \cos C = 0;\ \gamma \cos C - \alpha \cos A = 0;\ \text{and } \alpha \cos A - \beta \cos B = 0.$$

These meet where $\alpha \cos A = \beta \cos B = \gamma \cos C$, *i.e.* at the point

$$(\cos B \cos C,\ \cos C \cos A,\ \cos A \cos B).$$

Ex. 2. *Find the equation to the circle described upon the side BC of the triangle of reference as diameter.*

Any straight line through B is $\alpha - p\gamma = 0$, and one through C is $\alpha - q\beta = 0$.

These are at right angles if, by Art. 54,

$$1 - pq \cos A + p \cos B + q \cos C = 0.$$

Eliminating p and q, we have at their point of intersection

$$\beta\gamma + \alpha\,[\beta \cos B + \gamma \cos C - \alpha \cos A] = 0,$$

which is the equation required.

Ex. 3. *If p, q, r be the lengths of the perpendiculars drawn from the angular points of the triangle of reference upon any straight line, shew that*

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr \cos A - 2carp \cos B - 2abpq \cos C = 4\Delta^2,$$

where Δ is the area of the triangle, and that the length of the perpendicular from a point $(\alpha_0, \beta_0, \gamma_0)$ upon the line is

$$\frac{pa\alpha_0 + qb\beta_0 + rc\gamma_0}{2\Delta}.$$

Let $l\alpha + m\beta + n\gamma = 0$ be the equation to the straight line.

Then since the point A of the triangle of reference is $\left(\frac{2\Delta}{a}, 0, 0\right)$ we have, by Art. 59,

$$p = \frac{l \cdot \frac{2\Delta}{a}}{\sqrt{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C}} \quad \text{......(i)},$$

with similar expressions for q and r.

$$\therefore\ a^2p^2 + \ldots + \ldots - 2bcqr \cos A - \ldots - \ldots = 4\Delta^2.$$

Also the perpendicular from $(\alpha_0, \beta_0, \gamma_0)$

$$= \frac{l\alpha_0 + m\beta_0 + n\gamma_0}{\sqrt{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C}}$$

$$= \frac{ap\alpha_0 + bq\beta_0 + cr\gamma_0}{2\Delta}.$$

Cor. Since, by (i), $l : m : n :: pa : qb : rc$, the equation to the straight line is

$$pa\alpha+qb\beta+rc\gamma=0.$$

Ex. 4. *P, Q, R are points on the sides BC, CA, AB of a triangle ABC such that $BP.CQ.AR=PC.QA.RB$; shew that AP, BQ and CR are concurrent.* (Ceva's Theorem.)

If $BP.CQ.AR=-PC.QA.RB$, prove that P, Q, R are collinear. (Menelaus' Theorem.)

Let $\dfrac{BP}{PC}=\lambda$; $\dfrac{CQ}{QA}=\mu$; and $\dfrac{AR}{RB}=\nu$, so that, in the first case, $\lambda\mu\nu=1$.

The equation to AP is

$$\frac{\beta}{\gamma}=\frac{\text{perpendicular from } P \text{ on } CA}{\text{perpendicular from } P \text{ on } AB}=\frac{PC\sin C}{BP\sin B},$$

i.e. $\lambda\beta b=\gamma c$; so the equations of BQ and CR are $\mu\gamma c=\alpha a$ and $\nu\alpha a=\beta b$; on multiplying the equations together, we see that they are satisfied by common values of α, β, γ if $\lambda\mu\nu=1$, which is given.

In the second case, $\lambda\mu\nu=-1$. Here, for the point P,

$$\frac{\beta}{\gamma}=\frac{PC\sin C}{BP\sin B}=\frac{c}{\lambda b}.$$

Hence P is the point $(0, c, \lambda b)$. So Q is the point $(\mu c, 0, a)$ and R is $(b, \nu a, 0)$. These lie on a line if

$$\begin{vmatrix} 0, & c, & \lambda b \\ \mu c, & 0, & a \\ b, & \nu a, & 0 \end{vmatrix}=0,$$

i.e. if $\lambda\mu\nu abc+abc=0$, *i.e.* if $\lambda\mu\nu=-1$, which is given.

Ex. 5. *ABC is a triangle; AO, BO and CO are drawn to any point O and meet BC, CA and AB respectively in P, Q and R; QR and BC meet in P', RP and CA meet in Q', and PQ and AB in R'. Shew that P', Q', R' are collinear and that the ranges $[BPCP']$, $[CQAQ']$ and $[ARBR']$ are harmonic.*

Let the equation of AP be $m\beta-n\gamma=0$, let that of BQ be $n\gamma-l\alpha=0$.

These equations are clearly satisfied by the point whose coordinates are $\left(\dfrac{1}{l}, \dfrac{1}{m}, \dfrac{1}{n}\right)$, which is therefore the point O. Hence the equation to CR is $l\alpha-m\beta=0$.

The point Q is given by $\left.\begin{array}{r} n\gamma-l\alpha=0 \\ \beta=0 \end{array}\right\}$ ....(1),

and the point R by $\left.\begin{array}{r} l\alpha-m\beta=0 \\ \gamma=0 \end{array}\right\}$ ....(2).

As in Part I, Art. 82, any straight line through Q is given by

$$n\gamma-l\alpha+\lambda\beta=0,$$

and this passes through R if

$$\frac{-l}{l}=\frac{\lambda}{-m},\ \textit{i.e.}\text{ if }\lambda=m.$$

Hence the equation of QR is $-l\alpha+m\beta+n\gamma=0$

Hence P' is given by this and

$$\left.\begin{array}{r}-l\alpha+m\beta+n\gamma=0\\ \alpha=0\end{array}\right\}\ \ldots\ldots\ldots\ldots(3).$$

So Q' is given by

and

$$\left.\begin{array}{r}l\alpha-m\beta+n\gamma=0\\ \beta=0\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and R' is given by

and

$$\left.\begin{array}{r}l\alpha+m\beta-n\gamma=0\\ \gamma=0\end{array}\right\}.$$

Any straight line through P' is then given by

$$-l\alpha+m\beta+n\gamma+\mu\alpha=0,$$

and this passes through Q' if

$$\frac{\mu-l}{l}=\frac{n}{n},\ \textit{i.e.}\text{ if }\mu=2l.$$

Hence the equation of $P'Q'$ is $l\alpha+m\beta+n\gamma=0$.

This clearly passes through the point R', and hence P', Q' and R' are collinear.

For the equation of AP' we want an equation which is derived from equations (3) by adding some multiple of one of them to some multiple of the other; since AP' passes through A we also want an equation which does not contain α. Hence the required equation is obtained by multiplying the second of equations (3) by l and adding it to the first, *i.e.* the equation of AP' is $m\beta+n\gamma=0$. Also the equation of AP is $m\beta-n\gamma=0$ and by Art. 5, or as will be seen in Art. 61, the straight lines

$$\gamma=0,\quad m\beta-n\gamma=0,\quad \beta=0\text{ and }m\beta+n\gamma=0$$

form a harmonic pencil, *i.e.* $[BPCP']$ is a harmonic range. So also by symmetry $[CQAQ']$ and $[ARBR']$ are harmonic ranges.

The above example has been written out at full length; but in practice the work is much shorter. Thus from (1) and (2) the student would see on inspection that $-l\alpha+m\beta+n\gamma=0$ was satisfied both by equations (1) and by equations (2) and hence is the equation he wants for the line QR.

Similarly for the equation of $P'Q'$, since it is derivable from (3) and (4).

In working with trilinear coordinates attention must be given to considerations of symmetry, *e.g.* in the previous question the equation of RP was easily obtained from that of QR by a cyclical change of α, β, γ into β, γ, α and of l, m, n into m, n, l.

Ex. 6. *In the previous question shew that the lines AO, BQ' and CR' meet in a point U, the lines BO, CR' and AP' in a point V, and the lines CO, AP' and BQ' in a point W, and hence given the line $P'Q'R'$ determine the point O.*

The equations to AO, BQ', CR' are $m\beta-n\gamma=0$, $n\gamma+l\alpha=0$ and $l\alpha+m\beta=0$. These clearly meet in the point $(-mn, nl, lm)$. So for the other sets of three lines.

Given P', Q', R', let BQ' and CR' give U, CR' and AP' give V, and AP' and BQ' give W. Then AU, BV and CW give the point O.

The line $P'Q'R'$ is called the **polar** of the point O with respect to the triangle, and the point O is called the **pole** of the line $P'Q'R'$.

Ex. 7. *If two triangles ABC, $A'B'C'$ are such that the lines AA', BB', CC' meet in a point O, then the points of intersection of the corresponding sides, viz. BC and $B'C'$, CA and $C'A'$, AB and $A'B'$, lie on a straight line.*

Take ABC as triangle of reference and let the point O be (f, g, h). Then the point A' lies on the straight line $\frac{\beta}{g}=\frac{\gamma}{h}$, and hence its coordinates may be taken to be α', g, h. So B' is the point (f, β', h) and C' is (f, g, γ'). Hence the equation to the straight line $B'C'$ is thus

$$\alpha(\beta'\gamma'-gh)+\beta f(h-\gamma')+\gamma f(g-\beta')=0.$$

Its intersection, D, with BC is thus given by

$$\alpha=0 \text{ and } \frac{\beta}{g-\beta'}+\frac{\gamma}{h-\gamma'}=0.$$

Similarly for the other two intersections. They all clearly lie on the straight line

$$\frac{\alpha}{f-\alpha'}+\frac{\beta}{g-\beta'}+\frac{\gamma}{h-\gamma'}=0.$$

The two triangles ABC, $A'B'C'$ when they satisfy the condition of the question are said to be **in perspective**. The point O is called the **centre of perspective or homology**, and the straight line on which the intersections lie is called the **axis of perspective or homology**.

The above theorem is often expressed in the form **copolar triangles are also coaxial.** Their pole is the centre of perspective and their axis the axis of perspective.

More generally any figure of points P, Q, R, S, ... is said to be in perspective with a figure P', Q', R', S', ... if the straight lines PP', QQ', RR', SS', ... all meet in a point, the centre of perspective.

EXAMPLES.

1. Shew that the coordinates of the centroid of a triangle whose angular points are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$ are

$$\frac{\alpha_1+\alpha_2+\alpha_3}{3},\quad \frac{\beta_1+\beta_2+\beta_3}{3},\quad \text{and}\quad \frac{\gamma_1+\gamma_2+\gamma_3}{3}.$$

2. Shew that the lines

$$\alpha a^3+\beta b^3+\gamma c^3=0 \text{ and } \alpha\cos A+\beta\cos B+\gamma\cos C=0$$

are parallel.

3. Shew that the equation to the straight line joining the in-centre and circum-centre of the triangle of reference ABC is

$$\alpha(\cos B-\cos C)+\beta(\cos C-\cos A)+\gamma(\cos A-\cos B)=0,$$

that any point on it is $(m+n\cos A,\ m+n\cos B,\ m+n\cos C)$, and that it is perpendicular to the straight line

$$\alpha+\beta+\gamma=0.$$

4. Shew that the equation of the straight line through the circum-centre, centroid, nine-point centre, and ortho-centre of a triangle ABC is

$$\alpha\sin 2A\sin(B-C)+\beta\sin 2B\sin(C-A)+\gamma\sin 2C\sin(A-B)=0,$$

and that it is perpendicular to the straight line

$$\alpha\cos A+\beta\cos B+\gamma\cos C=0.$$

Shew also that the former of these straight lines passes through the four corresponding points for the triangle obtained by joining the middle points of the sides of the triangle ABC.

5. Shew that the equations of the straight line joining the vertex A of the triangle of reference to the circum-centre, centroid, nine-point centre and ortho-centre are respectively

$$\beta\cos C-\gamma\cos B=0, \qquad \beta\sin B-\gamma\sin C=0,$$
$$\beta\cos(A-B)-\gamma\cos(C-A)=0, \text{ and } \beta\cos B-\gamma\cos C=0,$$

and prove that they form a harmonic pencil.

6. Shew that the intersections of any side of a triangle with the external bisector of the opposite angle are collinear, and that their intersections with the tangents to the circum-circle at the opposite angles are collinear also.

7. Shew that the equation to the straight line which passes through the point $(\alpha', \beta', \gamma')$, and is perpendicular to the side BC of the triangle of reference, is

$$\begin{vmatrix} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \\ -1, & \cos C, & \cos B \end{vmatrix}=0.$$

8. Shew that the perpendiculars to the sides of a triangle through the middle points are concurrent.

9. Shew that the straight lines, which are parallel to the side BC of the triangle of reference and are twice as far from A as it is, are

$$2a\alpha+b\beta+c\gamma=0 \text{ and } 2a\alpha+3b\beta+3c\gamma=0.$$

10. Shew that the straight line $\beta=p\gamma$ is inclined to the side AB at an angle $\tan^{-1}\dfrac{\sin A}{p+\cos A}$.

11. If three straight lines through the vertices of a triangle are concurrent, shew that the three lines equally inclined with them to the bisectors are concurrent also.

[For $\beta-p\gamma=0$ and $\gamma-p\beta=0$ are corresponding lines.]

In the particular case where the first three lines are the medians, the second three lines are called the symmedians and their point of concurrence the symmedian point whose coordinates are (a, b, c).

12. If the straight lines $\frac{b\alpha}{c}=\frac{c\beta}{a}=\frac{a\gamma}{b}$ meet in a point O and the straight lines $\frac{c\alpha}{b}=\frac{a\beta}{c}=\frac{b\gamma}{a}$ in a point O', prove that the six angles OAB, OBC, OCA, $O'AC$, $O'BA$, $O'CB$ are all equal to the same angle ω, and that $\cot\omega=\cot A+\cot B+\cot C$.

[O and O' are known as the Brocard points of the triangle ABC, and ω as the Brocard angle.]

61. Cross-Ratios. The propositions of the preceding Chapter hold for trilinear coordinates just as for Cartesians.

Suppose that we have a pencil of four straight lines, passing through the point A of the triangle of reference, whose equations are

$$\beta=p_1\gamma,\ \beta=p_2\gamma,\ \beta=p_3\gamma \text{ and } \beta=p_4\gamma.$$

Let them be cut by a straight line parallel to the side AB in the points P_1, P_2, P_3, P_4 and let the straight line meet AC in O.

These points all have the same value for γ; hence the values of β for the points P_1, P_2, P_3, P_4 are proportional to p_1, p_2, p_3, p_4 and hence the values of OP_1, OP_2, OP_3, OP_4 are proportional to p_1, p_2, p_3, p_4.

$\therefore$ the cross-ratio of the pencil $=(P_1\,P_2\,P_3\,P_4)$

$$=\frac{(p_2-p_1)(p_4-p_3)}{(p_3-p_2)(p_1-p_4)}, \text{ as in Art. 4.}$$

Just as in Art. 5 it follows that the straight lines $\beta=0$, $\beta=p_1\gamma$, $\gamma=0$, and $\beta=p_2\gamma$ form a pencil whose cross-ratio is $\frac{p_1}{p_2}$, and that the four lines $\beta=0$, $\beta=p_1\gamma$, $\gamma=0$, $\beta=-p_1\gamma$ form a harmonic pencil.

Also, as in Art. 9, the straight lines

$$a\beta^2+2h\beta\gamma+b\gamma^2=0 \text{ and } a'\beta^2+2h'\beta\gamma+b'\gamma^2=0$$

are harmonically conjugate if

$$ab'+a'b=2hh'.$$

62. The cross-ratio of a pencil of four concurrent straight lines whose equations are

$$l_1\alpha+m_1\beta+n_1\gamma=0, \qquad l_2\alpha+m_2\beta+n_2\gamma=0,$$
$$l_3\alpha+m_3\beta+n_3\gamma=0, \text{ and } l_4\alpha+m_4\beta+n_4\gamma=0,$$

being equal to that of the four points in which these straight lines meet the side BC of the triangle of reference, *i.e.* $\alpha = 0$, is equal to the cross-ratio of the four straight lines

$$m_1\beta + n_1\gamma = 0, \qquad m_2\beta + n_2\gamma = 0,$$
$$m_3\beta + n_3\gamma = 0, \text{ and } m_4\beta + n_4\gamma = 0,$$

which join A to these four points of intersection.

63. *If* $P \equiv l_1\alpha + m_1\beta + n_1\gamma = 0$ *and* $Q \equiv l_2\alpha + m_2\beta + n_2\gamma = 0$, *the cross-ratio of the four straight lines given by*

$$P + \lambda_1 Q = 0,\ P + \lambda_2 Q = 0,\ P + \lambda_3 Q = 0, \textit{ and } P + \lambda_4 Q = 0$$

is
$$\frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_4)}.$$

For let X and Y be the perpendiculars from any point upon the two straight lines $P = 0$ and $Q = 0$, so that

$$X = \frac{l_1\alpha + m_1\beta + n_1\gamma}{\sqrt{l_1^2 + m_1^2 + n_1^2 - 2m_1n_1\cos A - 2n_1l_1\cos B - 2l_1m_1\cos C}}$$
$$= \frac{P}{C_1} \text{ (say)},$$

and
$$Y = \frac{Q}{C_2}.$$

The equations to the four given straight lines are thus

$$X = -\frac{\lambda_1 C_2 Y}{C_1}; \qquad X = -\frac{\lambda_2 C_2 Y}{C_1};$$
$$X = -\frac{\lambda_3 C_2 Y}{C_1}; \text{ and } X = -\frac{\lambda_4 C_2 Y}{C_1},$$

and then, as in Art. 61, the cross-ratio is as stated.

Cor. These straight lines will form a harmonic pencil if

$$(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_4) = 2\lambda_1\lambda_3 + 2\lambda_2\lambda_4.$$

64. Complete Quadrangles and Quadrilaterals.

Let P, Q, R and S be four points forming a quadrangle. Let PR and QS meet in A, PS and QR in B, and PQ and RS in C, so that A, B, C are the vertices of the quadrangle.

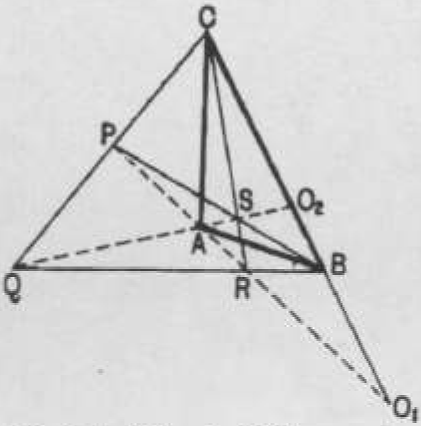
Take ABC as the triangle of reference.

Let the coordinates of *S* be *f*, *g*, *h* so that the equations of *BS* and *CS* are respectively

$$\frac{\alpha}{f}-\frac{\gamma}{h}=0 \quad \text{.....................(1)},$$

and

$$\frac{\alpha}{f}-\frac{\beta}{g}=0 \quad \text{.....................(2)}.$$



By Art. 15, *BQ*, *BA*, *BS* and *BC* form a harmonic pencil, so that the equation of *BR* is (Art. 61)

$$\frac{\alpha}{f}+\frac{\gamma}{h}=0 \quad \text{.....................(3)}.$$

By the same article $C[PASB]$ is harmonic, so that the equation of *CP* is

$$\frac{\alpha}{f}+\frac{\beta}{g}=0 \quad \text{.....................(4)}.$$

P is the point of intersection of (1) and (4), so that its coordinates are proportional to $f, -g, h$. *Q* similarly has its coordinates given by (3) and (4) proportional to $+f, -g, -h$, and, from (2) and (3), the coordinates of *R* are proportional to $f, g, -h$.

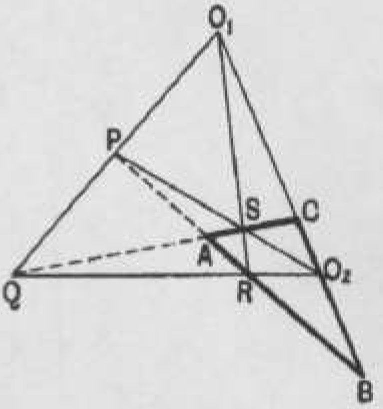
Ex. In addition to the equations found in the previous article, shew that the following lines have the equations set against them;

$$O_1RP,\ \frac{\beta}{g}+\frac{\gamma}{h}=0; \qquad O_2SQ,\ \frac{\beta}{g}-\frac{\gamma}{h}=0;$$

$$O_1Q,\ \frac{2\alpha}{f}+\frac{\beta}{g}+\frac{\gamma}{h}=0; \quad O_1S,\ -\frac{2\alpha}{f}+\frac{\beta}{g}+\frac{\gamma}{h}=0;$$

$$O_2P,\ \frac{2\alpha}{f}+\frac{\beta}{g}-\frac{\gamma}{h}=0; \quad O_2R,\ -\frac{2\alpha}{f}+\frac{\beta}{g}-\frac{\gamma}{h}=0.$$

65. Let PQ, QR, RS, and SP be four straight lines forming a quadrilateral. Let PQ and RS meet in O_1, and PS and QR in O_2. If the diagonals PR, QS, and O_1O_2 meet as in the figure, then ABC is the triangle formed by the diagonals of the complete quadrilateral. Take ABC as the triangle of reference.



Let the equation to PQ be $l\alpha+m\beta+n\gamma=0$, so that O_1 is given by this and $\alpha=0$. Hence the equation of AO_1 is $m\beta+n\gamma=0$.

Now the pencil $A[O_1CO_2B]$ is harmonic (Art. 15). Hence the equation of AO_2 is

$$m\beta-n\gamma=0.$$

Hence P is the point $\left.\begin{matrix} l\alpha+m\beta+n\gamma=0\\ \gamma=0\end{matrix}\right\}$

and O_2 is the point $\left.\begin{matrix} m\beta-n\gamma=0\\ \alpha=0\end{matrix}\right\}$.

$\therefore$ the equation to PS is $l\alpha+m\beta-n\gamma=0$.

So Q is the point $\left.\begin{matrix} l\alpha+m\beta+n\gamma=0\\ \beta=0\end{matrix}\right\}$.

Hence the equation to QO_2, *i.e.* QR, is $l\alpha-m\beta+n\gamma=0$.

Finally R is the point $\left.\begin{matrix} l\alpha-m\beta+n\gamma=0\\ \gamma=0\end{matrix}\right\}$,

and S is the point $\left.\begin{matrix} l\alpha+m\beta-n\gamma=0\\ \beta=0\end{matrix}\right\}$.

Hence the equation to RS is $-l\alpha+m\beta+n\gamma=0$.

The equations to the four straight **lines** forming the sides of the **quadrilateral** are therefore, when referred to the triangle formed by the three **diagonals** as the triangle of reference, of the form

$$l\alpha\pm m\beta\pm n\gamma=0.$$

Also the coordinates of the four **points** which are the angular points of the **quadrangle** are also, when referred to the triangle formed by the three **vertices** as the triangle of reference, of the form

$$\pm f,\ \pm g,\ \pm h.$$

It must be carefully noted that the triangle of reference is different in the two cases; the words in thick print may assist the memory; it will be noted that lines go with lines, and points with points.

Ex. In addition to the equations found in the previous article, shew that the following lines have equations as set against them

$$CP,\ l\alpha+m\beta=0;\quad CR,\ l\alpha-m\beta=0.$$
$$BQ,\ l\alpha+n\gamma=0;\quad BS,\ l\alpha-n\gamma=0.$$

66. Ex. *Shew that the middle points of the diagonals of a quadrilateral lie on a straight line.*

Using the notation and figure of the last article let U, V, W be the middle points of the diagonals O_1O_2, QS, and PR.

Since W is the middle point of PR it, with the point in which PR meets the line at infinity, divides PR harmonically. Hence CW and the line through C parallel to AB divide CP and CR harmonically.

The line through C parallel to AB is $a\alpha+b\beta=0$.

Also R being the point

$$\left.\begin{aligned} l\alpha-m\beta+n\gamma&=0\\ \gamma&=0\end{aligned}\right\}$$

the equation of CR is $l\alpha-m\beta=0$. So that of CP is $l\alpha+m\beta=0$. Hence if the equation of CW is $\alpha-\lambda\beta=0$, the four straight lines

$$l\alpha+m\beta=0,\quad \alpha-\lambda\beta=0,\quad l\alpha-m\beta=0,\ \text{and}\ a\alpha+b\beta=0$$

are a harmonic pencil.

Hence, by Art. 63, Cor.,

$$0=-\frac{m^2}{l^2}-\lambda\frac{b}{a}.$$

$\therefore\ \lambda=-\dfrac{am^2}{l^2b}$, and the equation to CW is $\dfrac{l^2}{a}\alpha+\dfrac{m^2}{b}\beta=0$.

$\therefore\ W$ is given by $\dfrac{l^2}{a}\alpha+\dfrac{m^2}{b}\beta=0$ and $\gamma=0$.

So U is given by $\dfrac{m^2}{b}\beta+\dfrac{n^2}{c}\gamma=0$ and $\alpha=0$. So for V.

All three points lie on the straight line

$$\frac{l^2}{a}\alpha+\frac{m^2}{b}\beta+\frac{n^2}{c}\gamma=0.$$

Areal Coordinates.

67. Another system of coordinates, which is often used, is one in which, instead of having the position of the point P (Art. 42) given by the perpendiculars on the sides of a fundamental triangle, we determine it by the ratios that the areas of the triangles PBC, PCA and PAB bear to the area of the triangle ABC. We shall denote these coordinates by the quantities x, y, z so that

$$\frac{x}{\Delta PBC}=\frac{y}{\Delta PCA}=\frac{z}{\Delta PAB}=\frac{1}{\Delta ABC}.$$

Now $\Delta PBC=\frac{1}{2}a\alpha$, etc., so that

$$x=\frac{a\alpha}{2\Delta},\quad y=\frac{b\beta}{2\Delta}\text{ and }z=\frac{c\gamma}{2\Delta},$$

where Δ is the area of the triangle ABC. Clearly we always have

$$x+y+z=1.$$

Any homogeneous equation in α, β, γ can be at once transferred to the corresponding areal equation by substituting $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ for α, β, γ respectively, and conversely.

68. The work of the previous chapter to a great extent is true for both trilinear and areal coordinates, and need not be repeated. Subjoined are the formulae and results which are different in areal coordinates from what they are in trilinear coordinates. They can be obtained independently or by substitution from the corresponding trilinear formulae.

When areal coordinates are implied we shall always use x, y, z as the coordinates, and $px+qy+rz=0$ as the standard form of the equation to a straight line.

I. The fundamental relation between the areal coordinates of any point is

$$x+y+z=1,$$

and the equation to the line at infinity is

$$x+y+z=0.\quad\text{[Arts. 43 and 56.]}$$

II. The area of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\Delta \times \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}. \qquad \text{[Art. 50.]}$$

III. The straight lines $px + qy + rz = 0$, $p'x + q'y + r'z = 0$ are perpendicular if

$$a^2pp' + b^2qq' + c^2rr' - bc\,(qr' + q'r)\cos A$$
$$- ca\,(rp' + r'p)\cos B - ab\,(pq' + p'q)\cos C = 0,$$

and they are parallel if

$$\begin{vmatrix} p, & q, & r \\ p', & q', & r' \\ 1, & 1, & 1 \end{vmatrix} = 0. \text{ [Arts. 54 and 57.]}$$

IV. The perpendicular distance of a given point (x', y', z') from the straight line $px + qy + rz = 0$ is [Art. 59]

$$\frac{(px' + qy' + rz') \times 2\Delta}{\sqrt{a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr\cos A - 2carp\cos B - 2abpq\cos C}}.$$

Since the areal coordinates of the angular points A, B, C of the triangle of reference are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, it follows that the perpendiculars from these angular points on the straight line are proportional to p, q, r. Also that, if p_1, q_1, r_1 are the actual perpendiculars from these angular points, the perpendicular from any point (x_0, y_0, z_0) is $p_1x_0 + q_1y_0 + r_1z_0$.

69. Just as the coordinates in trilinear coordinates are transformed into the corresponding areal coordinates by putting $\alpha = \frac{x}{a}$, $\beta = \frac{y}{b}$, and $\gamma = \frac{z}{c}$, so the constants in the trilinear equation of a straight line are transformed into the corresponding constants in the areal equation by putting $l = pa$, $m = qb$, and $n = rc$.

EXAMPLES.

1. Shew that the equation

$$\alpha^2+\beta^2+\gamma^2+\left(\frac{b}{c}+\frac{c}{b}\right)\beta\gamma+\left(\frac{c}{a}+\frac{a}{c}\right)\gamma\alpha+\left(\frac{a}{b}+\frac{b}{a}\right)\alpha\beta=0$$

gives only one finite line.

2. Find the equation to the third pair of straight lines which connects pairs of points given as the intersections of $\alpha^2=(m\beta+n\gamma)^2$ and $\beta^2=(m\gamma+n\alpha)^2$.

3. In the question of Art. 60, Ex. 5, if the point O moves on a fixed straight line, shew that its polar line $P'Q'R'$ touches a conic inscribed in the triangle ABC.

Shew also that BQ', CR', and AP meet in a point O_1, such that $(AOPO_1)$ is harmonic.

4. In areal coordinates shew that the straight lines

$$x\sin^2 A+y\sin^2 B+z\sin^2 C=0$$

and

$$x\cos^2 A+y\cos^2 B+z\cos^2 C=0$$

are both perpendicular to the line joining the centroid and ortho-centre of the triangle of reference.

5. The vertices of a triangle move on three concurrent lines and two of the sides pass each through a fixed point. Shew that the third side passes through a fixed point collinear with the two given fixed points.

6. The vertices of a triangle move along three fixed lines and two of the sides pass through two fixed points; find the envelope of the third side.

7. A variable line passes through a fixed point; find the locus of the point on it which forms a constant cross-ratio with the three points in which it cuts the triangle of reference.

8. ABC is a triangle and P, Q, R any points on the sides BC, CA and AB; QR and BC meet in D, RP and CA in E, and PQ and AB in F. If DF and BQ meet in G, and DE and CR in H, shew that EG and FH meet on BC at a point P', such that $[BPCP']$ is harmonic.

9. If a triangle ABC is circumscribed to the triangle PQR so that BC, CA, AB pass through the points P, Q, R respectively, shew that an infinite number of triangles can be drawn which are inscribed in the triangle PQR and also are circumscribed to the triangle ABC.

[Draw any line through A to meet PQ, PR in N and M and let BN meet QR in L; prove that LM passes through C.]

10. The perpendiculars from the vertices A, B, C of one triangle upon the sides $B'C'$, $C'A'$, $A'B'$ of a coplanar triangle are concurrent. Shew that the perpendiculars from A', B', C' on the sides BC, CA, AB respectively are also concurrent.

11. ABC is a triangle; AD, BE, and CF are the perpendiculars upon the opposite sides; EF meets BC in L. Shew that, if O is the

ortho-centre, OL is perpendicular to the line from A to the middle point of BC.

12. Points D, E, and F are taken on the sides of a triangle ABC such that AD, BE, and CF are concurrent and L, M, N are the middle points of EF, FD and DE respectively. Shew that AL, BM, and CN meet in a point.

[Let the equations to AD, BE, CF be $\beta - p\gamma = 0$, $\gamma - q\alpha = 0$ and $\alpha - r\beta = 0$, so that $pqr = 1$, and the equation to EF is $q\alpha - qr\beta - \gamma = 0$. Now (Art. 6), since L is the middle point of EF, AL and the line through A parallel to EF form a harmonic pencil with AB, AC. Also the equation to this parallel line is easily seen to be

$$q\beta\,(b + ar) + \gamma\,(a + cq) = 0.$$

Hence (Art. 61) the equation to AL is

$$q\beta\,(b + ar) = \gamma\,(a + cq).$$

So for BM, CN etc.]

13. Find the locus of a point P such that if D, E, F are the feet of the perpendiculars from it upon the sides of the triangle of reference ABC then AD, BE, and CF are concurrent.

14. A triangle PQR is circumscribed to a triangle ABC and is in perspective with it; prove that the lines joining A to the vertices of the triangle PQR are harmonic conjugates to AB and AC.

15. If parallels to $\alpha = 0$, $\beta = 0$ be drawn through the intersections of

$$l\alpha + m\beta + n\gamma = 0 \text{ and } l'\alpha + m'\beta + n'\gamma = 0,$$

shew that the four lines will form a harmonic pencil if

$$c^2\,(lm' + l'm) + 2abnn' = bc\,(nl' + n'l) + ca\,(mn' + m'n).$$

16. Shew that the cross-ratio of the four straight lines

$$\lambda\beta + \mu\gamma = 0,\quad \lambda_1\beta + \mu\gamma = 0,\quad \lambda\beta + \mu_1\gamma = 0 \text{ and } \lambda_1\beta + \mu_1\gamma = 0$$

is

$$\frac{\mu\mu_1\,(\lambda_1 - \lambda)^2}{(\lambda\mu - \lambda_1\mu_1)\,(\lambda\mu_1 - \lambda_1\mu)}.$$

17. Shew that the locus of a point which moves so that the straight lines drawn from it to four given non-collinear points form a pencil of given cross-ratio is a conic passing through the four given points.

[Take the four points as $(\pm f,\ \pm g,\ \pm h)$.]

18. Shew that in a complete quadrangle the three sides of the harmonic triangle are met by the sides of the quadrangle in six points, other than the vertices of the harmonic triangle, which lie by threes on four straight lines, and hence that these six points are the vertices of a complete quadrilateral whose diagonal triangle is that formed by the vertices of the quadrangle.

19. Each angle of the triangle formed by the diagonals of a complete quadrilateral is joined to the ends of the diagonal which does not pass through it; prove that these six lines pass, three by three, through four points.

CHAPTER III

TRILINEAR COORDINATES. EQUATIONS OF THE SECOND DEGREE

70. We shall take as the general equation the form

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0,$$

and shall often write it in the form $\phi(\alpha, \beta, \gamma) = 0$.

This form has the great advantage of agreeing with the general equation in Cartesian Coordinates; for, on putting $\gamma = 1$ and replacing α, β by x, y this equation becomes the standard Cartesian equation.

It has one disadvantage; the quantities a, b, c of this general equation may be mistaken for the lengths a, b, c of the triangle of reference. Whenever there is any possibility of mistake, the sides of the triangle of reference will be denoted by the symbols a_0, b_0, c_0.

71. *To find the equation to the tangent at any given point $(\alpha', \beta', \gamma')$ of the conic*

$$\phi(\alpha, \beta, \gamma) \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

As in Part I, Art. 373, the equation to the line joining the points $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$ lying on the curve is

$$\begin{aligned} &a(\alpha-\alpha')(\alpha-\alpha'') + b(\beta-\beta')(\beta-\beta'') + c(\gamma-\gamma')(\gamma-\gamma'') \\ &+ 2f(\beta-\beta')(\gamma-\gamma'') + 2g(\gamma-\gamma')(\alpha-\alpha'') + 2h(\alpha-\alpha')(\beta-\beta'') \\ &\qquad = \phi(\alpha, \beta, \gamma). \end{aligned}$$

For it is satisfied by the values $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$ and also by the values $\alpha = \alpha''$, $\beta = \beta''$, $\gamma = \gamma''$; also the terms of the second degree cancel on each side and hence the equation represents a straight line. Hence it is the required straight line.

Let now the point $(\alpha'', \beta'', \gamma'')$ move up to and ultimately coincide with the point $(\alpha', \beta', \gamma')$, and then the straight line becomes the tangent at $(\alpha', \beta', \gamma')$.

On putting $\alpha''=\alpha'$, $\beta''=\beta'$, and $\gamma''=\gamma'$ the equation becomes, since $\phi(\alpha',\beta',\gamma')=0$,

$$\mathbf{a}\alpha\alpha'+\mathbf{b}\beta\beta'+\mathbf{c}\gamma\gamma'+\mathbf{f}(\beta\gamma'+\beta'\gamma)$$
$$+\mathbf{g}(\gamma\alpha'+\gamma'\alpha)+\mathbf{h}(\alpha\beta'+\alpha'\beta)=\mathbf{0}.$$

This may be written in either of the forms

$$\alpha(a\alpha'+h\beta'+g\gamma')+\beta(h\alpha'+b\beta'+f\gamma')+\gamma(g\alpha'+f\beta'+c\gamma')=0,$$

or $$\alpha'(a\alpha+h\beta+g\gamma)+\beta'(h\alpha+b\beta+f\gamma)+\gamma'(g\alpha+f\beta+c\gamma)=0,$$

or again, in the notation of the Differential Calculus, it may be expressed as

$$\alpha\frac{d\phi}{d\alpha'}+\beta\frac{d\phi}{d\beta'}+\gamma\frac{d\phi}{d\gamma'}=0,$$

or $$\alpha'\frac{d\phi}{d\alpha}+\beta'\frac{d\phi}{d\beta}+\gamma'\frac{d\phi}{d\gamma}=0.$$

72. *To find the condition that the straight line*

$$l\alpha+m\beta+n\gamma=0\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

may touch the conic

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0\ldots\ldots(2).$$

The equation to the straight lines joining the angular point A of the triangle of reference to the intersections of the straight line and conic are given by eliminating α between the two equations, and hence it is

$$a(m\beta+n\gamma)^2+l^2(b\beta^2+c\gamma^2+2f\beta\gamma)-2l(g\gamma+h\beta)(m\beta+n\gamma)=0,$$

i.e. $$\beta^2[am^2+bl^2-2hlm]+2\beta\gamma[amn+fl^2-glm-hln]$$
$$+\gamma^2[an^2+cl^2-2gln]=0\ldots\ldots(3).$$

For, being derived from the equations of the straight line and conic, it is satisfied wherever they are, *i.e.* at their points of intersection; also it is satisfied at the point A, and breaks into two linear factors, real or imaginary, and is therefore two straight lines.

If the given line is a tangent, the two lines given by (3) coincide, and (3) is a perfect square. This is the case if

$$(amn+fl^2-glm-hln)^2=(am^2+bl^2-2hlm)(an^2+cl^2-2gln),$$

i.e. if

$$l^2(bc-f^2)+m^2(ca-g^2)+n^2(ab-h^2)+2mn(gh-af)$$
$$+2nl(hf-bg)+2lm(fg-ch)=0\ldots\ldots(4),$$

i.e. if

$$\mathbf{Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0} \ldots(5),$$

where A, B, C, F, G and H are the co-factors of a, b, c, f, g and h in the determinant

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

The equation (4) or (5) is called the **Tangential Equation of the Conic** whose Point Equation is (2). It may be written in the form

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots(6).$$

This form may also be obtained directly. For if

$$l\alpha + m\beta + n\gamma = 0$$

is the equation to the tangent at the point $(\alpha', \beta', \gamma')$, then, comparing it with the equation to the tangent in the previous article, we have

$$\frac{a\alpha' + h\beta' + g\gamma'}{l} = \frac{h\alpha' + b\beta' + f\gamma'}{m} = \frac{g\alpha' + f\beta' + c\gamma'}{n} = \lambda \text{ (say)},$$

$$\begin{aligned} \therefore\ a\alpha' + h\beta' + g\gamma' - \lambda l &= 0, \\ h\alpha' + b\beta' + f\gamma' - \lambda m &= 0, \\ g\alpha' + f\beta' + c\gamma' - \lambda n &= 0. \end{aligned}$$

Also $$l\alpha' + m\beta' + n\gamma' = 0.$$

Eliminating α', β', γ', and λ from these equations we have (6).

Some particular cases of the tangential equation are as follows:

Circum-conic, $L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0$; Tangential Equation

$$L^2l^2 + M^2m^2 + N^2n^2 - 2MNmn - 2NLnl - 2LMlm = 0,$$

i.e. $$\sqrt{Ll} + \sqrt{Mm} + \sqrt{Nn} = 0.$$

In-conic, $\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0$; $Lmn + Mnl + Nlm = 0$.

Self-conjugate conic $L\alpha^2 + M\beta^2 + N\gamma^2 = 0$; $\frac{l^2}{L} + \frac{m^2}{M} + \frac{n^2}{N} = 0$.

Conic $\beta\gamma - k\alpha^2 = 0$; $l^2 - 4kmn = 0$.

73. It is easily seen that the Point-Equation of the conic is obtained from the Tangential or Line-Equation of the conic in precisely the same way as the Tangential Equation is obtained from the Point-Equation.

For, starting with equation (5) of the previous Article, we have

$$\begin{aligned} BC - F^2 &= (ca - g^2)(ab - h^2) - (gh - af)^2 \\ &= a\left[abc + 2fgh - af^2 - bg^2 - ch^2\right] = a\Delta. \end{aligned}$$

Similarly $CA - G^2 = b\Delta$, and $AB - H^2 = c\Delta$.

Also

$$\begin{aligned} GH - AF &= (hf - bg)(fg - ch) - (bc - f^2)(gh - af) \\ &= f\left[abc + 2fgh - af^2 - bg^2 - ch^2\right] = f\Delta. \end{aligned}$$

Similarly $HF - BG = g\Delta$, and $FG - CH = h\Delta$.

Thus, if to the coefficients of (5) we apply the same process by which the coefficients of (4) are derived from those of (2), we arrive at (2).

This may be verified independently by finding, as in Part I, Art. 437, the envelope of (1) under condition (5).

For, eliminating l, we have

$$\begin{aligned} m^2\left[A\beta^2 + B\alpha^2 - 2H\alpha\beta\right] + 2mn\left[A\beta\gamma + F\alpha^2 - G\alpha\beta - H\gamma\alpha\right] & \\ + n^2\left[A\gamma^2 + C\alpha^2 - 2G\gamma\alpha\right] &= 0, \end{aligned}$$

the envelope of which is

$$\begin{aligned} &(A\beta\gamma + F\alpha^2 - G\alpha\beta - H\gamma\alpha)^2 \\ &\qquad = (A\beta^2 + B\alpha^2 - 2H\alpha\beta)(A\gamma^2 + C\alpha^2 - 2G\gamma\alpha), \end{aligned}$$

i.e.

$$\alpha^2(BC - F^2) + \ldots + \ldots + 2(GH - AF)\beta\gamma + \ldots + \ldots = 0 \ldots (6),$$

i.e., by the relations proved above,

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0,$$

which is (2).

Clearly the coefficients in equation (6) are derived from those in (5) by the same process by which the coefficients in equation (4) are obtained from those in (1).

74. *Condition that the conic may be a parabola.*

If the conic be a parabola it touches the line at infinity $a_0\alpha + b_0\beta + c_0\gamma = 0$. Hence a_0, b_0, c_0 must satisfy the tangential equation of Art. 72, *i.e.*

$$Aa_0^2 + Bb_0^2 + Cc_0^2 + 2Fb_0c_0 + 2Gc_0a_0 + 2Ha_0b_0 = 0,$$

or

$$(bc - f^2)\,a_0^2 + \ldots + \ldots + 2\,(gh - af)\,b_0c_0 + \ldots + \ldots = 0.$$

The condition that the conic may be an ellipse or a hyperbola may be found by finding its intersections with the line at infinity, and introducing the condition that they may be both imaginary or both real. On substituting a_0, b_0, c_0 for l, m, n in equation (3) of Art. 72, we should find that the conic is an ellipse or hyperbola according as

$$Aa_0^2 + Bb_0^2 + Cc_0^2 + 2Fb_0c_0 + 2Gc_0a_0 + 2Ha_0b_0$$

is positive or negative.

75. *Polar of a given point.*

Just as in Part I, Art. 375, it could now be shewn that the polar of a point $(\alpha', \beta', \gamma')$ with respect to $\phi\,(\alpha, \beta, \gamma) = 0$ is

$$\alpha\,(a\alpha' + h\beta' + g\gamma') + \beta\,(h\alpha' + b\beta' + f\gamma') + \gamma\,(g\alpha' + f\beta' + c\gamma') = 0 \quad \ldots(1),$$

i.e. the polar of the point $(\alpha', \beta', \gamma')$ is of the same form as the tangent at the point $(\alpha', \beta', \gamma')$ when the latter is on the curve.

This may be written either as

$$\alpha\frac{d\phi}{d\alpha'} + \beta\frac{d\phi}{d\beta'} + \gamma\frac{d\phi}{d\gamma'} = 0,$$

or as

$$\alpha'\frac{d\phi}{d\alpha} + \beta'\frac{d\phi}{d\beta} + \gamma'\frac{d\phi}{d\gamma} = 0.$$

The polar of $(\alpha', \beta', \gamma')$ will therefore pass through the point $(\alpha'', \beta'', \gamma'')$, *i.e.* these two points will be conjugate, if α'', β'', γ'' satisfies equation (1), *i.e.* if

$$a\alpha'\alpha'' + b\beta'\beta'' + c\gamma'\gamma'' + f(\beta'\gamma'' + \beta''\gamma') + g\,(\gamma'\alpha'' + \gamma''\alpha') + h\,(\alpha'\beta'' + \alpha''\beta') = 0.$$

76. *Pole of a given line.*

If the given line be

$$l\alpha + m\beta + n\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

then, if its pole be $(\alpha', \beta', \gamma')$, this equation must be the same as (1) in the previous article.

Hence, for some value of λ, we have

$$a\alpha' + h\beta' + g\gamma' = l\lambda,$$

$$h\alpha' + b\beta' + f\gamma' = m\lambda,$$

and

$$g\alpha' + f\beta' + c\gamma' = n\lambda.$$

If A, B, C etc. be defined as in Art. 72, we thus have

$$\frac{\alpha'}{Al + Hm + Gn} = \frac{\beta'}{Hl + Bm + Fn} = \frac{\gamma'}{Gl + Fm + Cn} = \frac{\lambda}{\Delta}.$$

Hence the pole of (1) is

$$(Al + Hm + Gn,\ Hl + Bm + Fn,\ Gl + Fm + Cn).$$

This lies on the straight line

$$l'\alpha + m'\beta + n'\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

i.e. the lines (1) and (2) are conjugate, if

$$l'(Al + Hm + Gn) + m'(Hl + Bm + Fn) + n'(Gl + Fm + Cn) = 0,$$

i.e. if

$$All' + Bmm' + Cnn' + F(mn' + m'n) + G(nl' + n'l) + H(lm' + l'm) = 0.$$

77. *To shew that the diameter of the conic*

$$\phi(\alpha, \beta, \gamma) \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0,$$

which bisects chords parallel to the line $l\alpha + m\beta + n\gamma = 0$, *is*

$$\frac{d\phi}{d\alpha}(mc - nb) + \frac{d\phi}{d\beta}(na - lc) + \frac{d\phi}{d\gamma}(lb - ma) = 0.$$

The diameter required is the one which is conjugate to the given chords, and is therefore the polar of the point in which the given lines meet the line at infinity. This point is given by the intersection with $a\alpha + b\beta + c\gamma = 0$ of the given line, and is $(mc - nb,\ na - lc,\ lb - ma)$. The polar of this point (by Art. 75) is

$$(mc - nb)(a\alpha + h\beta + g\gamma) + (na - lc)(h\alpha + b\beta + f\gamma) + (lb - ma)(g\alpha + f\beta + c\gamma) = 0.$$

78. *Coordinates of the centre of the conic.*

Let $(\alpha_1, \beta_1, \gamma_1)$ be the centre. Then its polar is the line at infinity, and therefore is

$$a_0\alpha + b_0\beta + c_0\gamma = 0,$$

where a_0, b_0, c_0 are the sides of the triangle of reference.

Comparing this with equation (1) of Art. 75, we have

$$\frac{a\alpha_1 + h\beta_1 + g\gamma_1}{a_0} = \frac{h\alpha_1 + b\beta_1 + f\gamma_1}{b_0} = \frac{g\alpha_1 + f\beta_1 + c\gamma_1}{c_0} \ldots (1),$$

equations to give the required coordinates.

Aliter. The centre is the pole of the line

$$a_0\alpha + b_0\beta + c_0\gamma = 0.$$

Hence, as in Art. 76, its coordinates are

$$(Aa_0 + Hb_0 + Gc_0,\ Ha_0 + Bb_0 + Fc_0,\ Ga_0 + Fb_0 + Cc_0).$$

Or these coordinates can be obtained by solving equations (1).

Particular Cases.

Circum-conic, $L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0$;
centre,
$[L(Mb_0 + Nc_0 - La_0),\ M(Nc_0 + La_0 - Mb_0),\ N(La_0 + Mb_0 - Nc_0)]$.
In-conic, $\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0$;
centre, $[Mc_0 + Nb_0,\ Na_0 + Lc_0,\ Lb_0 + Ma_0]$.
Self-conjugate conic, $L\alpha^2 + M\beta^2 + N\gamma^2 = 0$;
centre, $\left(\frac{a_0}{L}, \frac{b_0}{M}, \frac{c_0}{N}\right)$.
Conic, $\beta\gamma - k\alpha^2 = 0$;
centre, $(-a_0,\ 2kc_0,\ 2kb_0)$.

79. *Equation to the tangents from a given point.*

Just as in Part I, Art. 389, the equation to the pair of tangents from $(\alpha', \beta', \gamma')$ to the conic $\phi(\alpha, \beta, \gamma)$ is

$$\phi(\alpha, \beta, \gamma) \times \phi(\alpha', \beta', \gamma') = u^2 \ldots\ldots\ldots\ldots (1),$$

where

$$u \equiv \alpha(a\alpha' + h\beta' + g\gamma') + \beta(h\alpha' + b\beta' + f\gamma') + \gamma(g\alpha' + f\beta' + c\gamma').$$

On simplification, this equation becomes, with the notation of Art. 72,

$$\alpha^2[C\beta'^2 + B\gamma'^2 - 2F\beta'\gamma'] + \ldots + \ldots$$
$$+ 2\beta\gamma[-F\alpha'^2 - A\beta'\gamma' + H\gamma'\alpha' + G\alpha'\beta'] + \ldots + \ldots = 0.$$

Director Circle. By Art. 54, Cor., these straight lines are at right angles if

$$(C\beta'^2 + B\gamma'^2 - 2F\beta'\gamma') + \ldots + \ldots$$
$$+ 2\left[F\alpha'^2 + A\beta'\gamma' - H\gamma'\alpha' - G\alpha'\beta'\right]\cos A + \ldots + \ldots = 0.$$

Hence the locus of $(\alpha', \beta', \gamma')$, *i.e.* the Director Circle, is

$$\alpha^2\left[B + C + 2F\cos A\right] + \ldots + \ldots$$
$$+ 2\beta\gamma\left[A\cos A - F - G\cos C - H\cos B\right] + \ldots + \ldots = 0 \ldots (2).$$

Particular Cases.

Circum-conic, $L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0$;

director circle,

$$\alpha^2(M^2 + N^2 - 2MN\cos A) + \ldots + \ldots$$
$$+ 2\beta\gamma\left[L(L\cos A + M\cos B + N\cos C) + MN\right] + \ldots + \ldots = 0.$$

In-conic, $\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0$;

director circle,

$$L\cos A\alpha^2 + \ldots + \ldots - \beta\gamma\left[L + M\cos C + N\cos B\right] - \ldots - \ldots = 0.$$

Self-conjugate conic, $L\alpha^2 + M\beta^2 + N\gamma^2 = 0$;

director circle,

$$L(M + N)\alpha^2 + \ldots + \ldots + 2MN\cos A \,.\, \beta\gamma + \ldots + \ldots = 0.$$

Conic, $\beta\gamma - k\alpha^2 = 0$;

director circle,

$$-4k\cos A\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma(2k + \cos A) + 4k\gamma\alpha\cos C + 4k\alpha\beta\cos B = 0.$$

80. *Condition that the conic may be a rectangular hyperbola.*

On changing the general equation to Cartesian Coordinates, as in Art. 46, the coefficients of x^2 and y^2 are respectively

$$a\cos^2\omega_1 + b\cos^2\omega_2 + c\cos^2\omega_3 + 2f\cos\omega_2\cos\omega_3$$
$$+ 2g\cos\omega_3\cos\omega_1 + 2h\cos\omega_1\cos\omega_2,$$

and

$$a\sin^2\omega_1 + b\sin^2\omega_2 + c\sin^2\omega_3 + 2f\sin\omega_2\sin\omega_3$$
$$+ 2g\sin\omega_3\sin\omega_1 + 2h\sin\omega_1\sin\omega_2.$$

Hence, by Part I, Art. 358, it represents a rectangular hyperbola if

$$a + b + c + 2f\cos(\omega_2 - \omega_3) + 2g\cos(\omega_3 - \omega_1) + 2h\cos(\omega_1 - \omega_2) = 0,$$

or, by Art. 46, if

$$a + b + c - 2f\cos A - 2g\cos B - 2h\cos C = 0.$$

81. *Condition that the general equation may represent two straight lines.*

Just as in Part I, Art. 116, the equation breaks up into linear factors, and therefore represents two straight lines, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

82. *Equation to the asymptotes of the general conic.*

The asymptotes are the tangents drawn from the centre to the conic, and are therefore given by the first equation of Art. 79, where $(\alpha', \beta', \gamma')$ are given by

$$\frac{a\alpha' + h\beta' + g\gamma'}{a_0} = \frac{h\alpha' + b\beta' + f\gamma'}{b_0} = \frac{g\alpha' + f\beta' + c\gamma'}{c_0} = \lambda,$$

and hence

$$\frac{\alpha'}{Aa_0 + Hb_0 + Gc_0} = \frac{\beta'}{Ha_0 + Bb_0 + Fc_0} = \frac{\gamma'}{Ga_0 + Fb_0 + Cc_0} = \frac{\lambda}{\Delta},$$

with the notation of Art. 72.

Then

$$\phi(\alpha', \beta', \gamma') = \alpha' . a_0\lambda + \beta' . b_0\lambda + \gamma' . c_0\lambda$$

$$= \frac{\lambda^2}{\Delta}[a_0(Aa_0 + Hb_0 + Gc_0) + b_0(Ha_0 + Bb_0 + Fc_0)$$
$$+ c_0(Ga_0 + Fb_0 + Cc_0)]$$

$$= \frac{\lambda^2}{\Delta}[Aa_0^2 + Bb_0^2 + Cc_0^2 + 2Fb_0c_0 + 2Gc_0a_0 + 2Ha_0b_0].$$

Also

$$u = \alpha[a_0\lambda] + \beta[b_0\lambda] + \gamma[c_0\lambda] = \lambda[a_0\alpha + b_0\beta + c_0\gamma].$$

Hence equation (1) of Art. 79 becomes

$$\phi(\alpha, \beta, \gamma)[Aa_0^2 + Bb_0^2 + Cc_0^2 + 2Fb_0c_0 + 2Gc_0a_0 + 2Ha_0b_0]$$
$$= \Delta(a_0\alpha + b_0\beta + c_0\gamma)^2,$$

which is therefore the required equation of the asymptotes.

83. *Equation to the circle circumscribing the triangle of reference.*

Let P be any point $(\alpha', \beta', \gamma')$ so situated that β' is negative and α', γ' both positive. Draw PL, PM, and PN perpendicular to BC, CA, AB respectively, so that $\angle MPN = A$, $\angle NPL = \pi - B$, and $\angle LPM = C$.

The areas of the triangles MPN, NPL, LPM are then respectively

$$\tfrac{1}{2}PM.PN\sin A,\ \tfrac{1}{2}PN.PL\sin B, \text{ and } \tfrac{1}{2}PL.PM\sin C,$$

i.e. $$-\tfrac{1}{2}\beta'\gamma'\sin A,\ \tfrac{1}{2}\gamma'\alpha'\sin B, \text{ and } -\tfrac{1}{2}\alpha'\beta'\sin C.$$

$$\therefore\ \triangle LMN=\triangle PMN+\triangle PML-\triangle PNL$$
$$=-\tfrac{1}{2}[\beta'\gamma'\sin A+\gamma'\alpha'\sin B+\alpha'\beta'\sin C].$$

[The same result will be found to be true for the magnitude of the area of the triangle for all positions of P.]

If the point P lies on the circumcircle of the triangle ABC, the feet L, M, N of the perpendiculars lie on a straight line, viz. the Pedal or Simson line of P, and hence the area of the triangle LMN is zero.

Hence, in this case,

$$\beta'\gamma'\sin A+\gamma'\alpha'\sin B+\alpha'\beta'\sin C=0,$$

i.e., the equation to the circumcircle is

$$\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C=0,$$

or $$a_0\beta\gamma+b_0\gamma\alpha+c_0\alpha\beta=0.$$

As in Art. 72 the tangential equation of the circumcircle is easily seen to be

$$\sqrt{la_0}+\sqrt{mb_0}+\sqrt{nc_0}=0.$$

84. We have seen, in Part I, Arts. 387 and 388, that, if $S=0$ be the equation to any circle, the equation to any other circle is of the form $S=\lambda_1 u$, where u is of the first degree, and that the equation to any concentric circle is of the form $S=\mu_1$, where μ_1 is a constant.

Using Art. 44, in order to make the equations homogeneous, we see that the equation to any circle whatever is

$$S=(l\alpha+m\beta+n\gamma)(a_0\alpha+b_0\beta+c_0\gamma),$$

and that the equation to any concentric circle is

$$S=\mu(a_0\alpha+b_0\beta+c_0\gamma)^2,$$

where l, m, n and μ are any constants whatever, and $S=0$ is the equation to any circle.

85. *To find the condition that the general equation of the second degree*

$$\phi(\alpha, \beta, \gamma) \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \ldots (1)$$

may represent a circle.

Taking $S = 0$ as the circum-circle, we see that the equation to any circle is of the form

$$a_0\beta\gamma + b_0\gamma\alpha + c_0\alpha\beta = (l\alpha + m\beta + n\gamma)(a_0\alpha + b_0\beta + c_0\gamma).$$

Comparing this with equation (1), we have

$$\frac{a}{la_0} = \frac{b}{mb_0} = \frac{c}{nc_0} = \frac{2f}{mc_0 + nb_0 - a_0} = \frac{2g}{na_0 + lc_0 - b_0}$$
$$= \frac{2h}{lb_0 + ma_0 - c_0} = \lambda \text{ (say)}.$$

The required condition is therefore

$$bc_0^2 + cb_0^2 - 2fb_0c_0 = ca_0^2 + ac_0^2 - 2gc_0a_0 = ab_0^2 + ba_0^2 - 2ha_0b_0;$$

for each of these quantities is equal to $a_0b_0c_0\lambda$.

86. *The equation to any circle being*

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma),$$

to shew that $l = \frac{t_1^2}{bc}$, $m = \frac{t_2^2}{ca}$, and $n = \frac{t_3^2}{ab}$, *where* t_1, t_2, t_3 *are the lengths of the tangents drawn from the vertices* A, B, C *of the triangle of reference to the circle, and* a, b, c *are the lengths of the sides of the triangle of reference.*

Let the circle meet the side BC of the triangle of reference in the points A_1 and A_2. Then these points are given, on putting $\alpha = 0$ in the equation to the circle, by

$$mb\beta^2 + (mc + nb - a)\beta\gamma + nc\gamma^2 = 0 \ldots\ldots\ldots(1).$$

If x be the distance of the point A_1 from B, then

$$\frac{\beta}{\gamma} = \frac{A_1C \sin C}{BA_1 \sin B} = \frac{(a - x) \,.\, c}{xb}.$$

Hence (1) gives

$$mbc^2(a - x)^2 + (mc + nb - a)\,bcx(a - x) + nb^2cx^2 = 0,$$

i.e.
$$x^2 - x[mc - nb + a] + mca = 0.$$

The roots of this quadratic are BA_1 and BA_2; hence mca = product of the roots $= BA_1 \,.\, BA_2 = t_2^2$.

So for the others.

$$\therefore\ l = \frac{t_1^2}{bc},\quad m = \frac{t_2^2}{ca},\ \text{and}\ n = \frac{t_3^2}{ab}.$$

Cor. The radical axis of this circle and the circum-circle of the triangle of reference is

$$\frac{t_1^2}{bc}\alpha + \frac{t_2^2}{ca}\beta + \frac{t_3^2}{ab}\gamma = 0.$$

87. From the preceding article we can easily write down the equations to some important circles.

In-circle. Here $t_1 = s - a$, $t_2 = s - b$, and $t_3 = s - c$.

Hence its equation is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta$$
$$= \left[\frac{(s-a)^2}{bc}\alpha + \frac{(s-b)^2}{ca}\beta + \frac{(s-c)^2}{ab}\gamma\right][a\alpha + b\beta + c\gamma],$$

i.e.

$$a^2(s-a)^2\alpha^2 + \dots + \dots + bc\beta\gamma\left[(s-b)^2 + (s-c)^2 - a^2\right] + \dots + \dots = 0.$$

Now

$$(s-b)^2 + (s-c)^2 + 2(s-b)(s-c) = (s-b+s-c)^2 = a^2.$$

Hence this equation is

$$a^2(s-a)^2\alpha^2 + b^2(s-b)^2\beta^2 + c^2(s-c)^2\gamma^2 - 2bc(s-b)(s-c)\beta\gamma$$
$$- 2ca(s-c)(s-a)\gamma\alpha - 2ab(s-a)(s-b)\alpha\beta = 0,$$

i.e. $$\sqrt{a(s-a)\alpha} + \sqrt{b(s-b)\beta} + \sqrt{c(s-c)\gamma} = 0,$$

or $$\sqrt{\alpha\cos^2\frac{A}{2}} + \sqrt{\beta\cos^2\frac{B}{2}} + \sqrt{\gamma\cos^2\frac{C}{2}} = 0.$$

The tangential equation, by Art. 72, is found to be

$$mn\cos^2\frac{A}{2} + nl\cos^2\frac{B}{2} + lm\cos^2\frac{C}{2} = 0.$$

Escribed-circle opposite to A. Here

$$t_1 = s,\quad t_2 = s - c,\ \text{and}\ t_3 = s - b.$$

Hence its equation is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = \left[\frac{s^2}{bc}\alpha + \frac{(s-c)^2}{ca}\beta + \frac{(s-b)^2}{ab}\gamma\right][a\alpha + b\beta + c\gamma],$$

i.e.

$$a^2s^2\alpha^2+b^2(s-c)^2\beta^2+c^2(s-b)^2\gamma^2+\beta\gamma\left[(s-b)^2+(s-c)^2-a^2\right]bc$$
$$+\gamma\alpha\left[(s-b)^2+s^2-b^2\right]ca+\alpha\beta\left[(s-c)^2+s^2-c^2\right]ab=0,$$

i.e., as before,

$$a^2s^2\alpha^2+b^2(s-c)^2\beta^2+c^2(s-b)^2\gamma^2-2bc(s-b)(s-c)\beta\gamma$$
$$+2cas(s-b)\gamma\alpha+2abs(s-c)\alpha\beta=0,$$

i.e. $$\sqrt{-as\alpha}+\sqrt{b(s-c)\beta}+\sqrt{c(s-b)\gamma}=0,$$

or $$\sqrt{-\alpha\cos^2\frac{A}{2}}+\sqrt{\beta\sin^2\frac{B}{2}}+\sqrt{\gamma\sin^2\frac{C}{2}}=0.$$

The tangential equation is (Art. 72)

$$-mn\cos^2\frac{A}{2}+nl\sin^2\frac{B}{2}+lm\sin^2\frac{C}{2}=0.$$

Nine-point circle. This circle passes through the middle points of the sides of the triangle, and through the feet of the perpendiculars from the vertices upon the opposite sides. Hence $t_1^2=\frac{b}{2}\cdot c\cos A$, etc., so that the equation is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=\tfrac{1}{2}\left[\alpha\cos A+\beta\cos B+\gamma\cos C\right]\left[a\alpha+b\beta+c\gamma\right],$$

i.e.

$$a\cos A\,.\,\alpha^2+b\cos B\,.\,\beta^2+c\cos C\,.\,\gamma^2-a\beta\gamma-b\gamma\alpha-c\alpha\beta=0.$$

The tangential equation is (Art. 72)

$$l^2b^2c^2(b^2-c^2)^2+\ldots+\ldots+2mna^2bc\left[a^4-(b^2c^2+c^2a^2+a^2b^2)\right]$$
$$+\ldots+\ldots=0.$$

This can be written in the form

$$a\sqrt{\frac{m}{b}+\frac{n}{c}}+b\sqrt{\frac{n}{c}+\frac{l}{a}}+c\sqrt{\frac{l}{a}+\frac{m}{b}}=0.$$

Self-conjugate circle. If AD, BE, CF be the perpendiculars upon the opposite sides meeting in O, this circle has centre O and radius $\sqrt{OA\,.\,OD}$, it being only real when one of the angles of the triangle is obtuse.

Now $\sqrt{OA\,.\,OD}=\sqrt{-2R\cos A\,.\,2R\cos B\cos C}$.

$$\therefore\ t_1^2=OA^2-(\text{rad.})^2=OA^2-OA\,.\,OD$$
$$=4R^2\cos^2A+4R^2\cos A\cos B\cos C=4R^2\cos A\sin B\sin C$$
$$=bc\cos A,$$

and so for the others. Hence the required equation is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (\alpha\cos A + \beta\cos B + \gamma\cos C)(a\alpha + b\beta + c\gamma),$$

i.e. $$a\cos A\alpha^2 + b\cos B\beta^2 + c\cos C\gamma^2 = 0,$$

i.e. $$\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C = 0.$$

The tangential equation is (Art. 72)

$$\frac{l^2}{\sin 2A} + \frac{m^2}{\sin 2B} + \frac{n^2}{\sin 2C} = 0.$$

Circle on BC as diameter. This circle goes through E and F. Hence $t_1^2 = AC \,.\, AE = b \,.\, c\cos A$. Also $t_2^2 = 0$ and $t_3^2 = 0$. Hence the equation required is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma) \,.\, \alpha\cos A,$$

i.e. $$\alpha^2\cos A - \beta\gamma - \gamma\alpha\cos C - \alpha\beta\cos B = 0.$$

(See Art. 60, Ex. 2.)

88. Ex. 1. *Shew that the equation to the circle which passes through the centre of the three escribed circles of the triangle of reference is*

$$a\alpha^2 + b\beta^2 + c\gamma^2 + (a+b+c)(\beta\gamma + \gamma\alpha + \alpha\beta) = 0,$$

and that of the circle through the in-centre and two of the e-centres is

$$a\alpha^2 + b\beta^2 - c\gamma^2 - (a+b-c)(\beta\gamma + \gamma\alpha - \alpha\beta) = 0.$$

The equation to any circle is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (a\alpha + b\beta + c\gamma)(l\alpha + m\beta + n\gamma).$$

If this passes through the e-centres whose coordinates are proportional to $(1, 1, -1)$, $(1, -1, 1)$ and $(-1, 1, 1)$, we have

$$l + m - n = \frac{-a-b+c}{a+b-c} = -1,$$

$$l - m + n = -1,$$

and $$-l + m + n = -1.$$

$$\therefore\ l = m = n = -1,$$

and the equation is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (a\alpha + b\beta + c\gamma)(\alpha + \beta + \gamma) = 0.$$

Similarly, if the circle passes through the in-centre and the two e-centres opposite the points A and B, whose coordinates are proportional to $(1, 1, 1)$, $(-1, 1, 1)$ and $(1, -1, 1)$, we have $l = 1$, $m = 1$, $n = -1$, and hence as before.

Ex. 2. *Shew that the nine-point circle, the self-conjugate circle, and the circum-circle of a triangle intersect in the same points.*

The radical axis of these three circles taken in pairs is, by the previous article,

$$\alpha\cos A + \beta\cos B + \gamma\cos C = 0.$$

Hence they all three meet where this line meets either of them.

Ex. 3. *Shew that the tangential equation of the circle, whose centre is the point* $(\alpha_1, \beta_1, \gamma_1)$ *and whose radius is* r, *is*

$$(l\alpha_1+m\beta_1+n\gamma_1)^2=r^2[l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C],$$

and hence that the point-equation is

$$\alpha^2[r^2\sin^2 A-(\beta_1^2+\gamma_1^2+2\beta_1\gamma_1\cos A)]+\ldots+\ldots$$
$$+2\beta\gamma[r^2\sin B\sin C+\beta_1\gamma_1+\gamma_1\alpha_1\cos C+\alpha_1\beta_1\cos B-\alpha_1^2\cos A]$$
$$+\ldots+\ldots=0.$$

If $l\alpha+m\beta+n\gamma=0$ is the equation to any tangent, the perpendicular upon it from the centre $(\alpha_1, \beta_1, \gamma_1)$ is equal to r, and hence, by Art. 59,

$$(l\alpha_1+m\beta_1+n\gamma_1)^2=r^2(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C),$$

the equation required,

i.e. $$l^2(\alpha_1^2-r^2)+\ldots+\ldots+2mn(\beta_1\gamma_1+r^2\cos A)+\ldots+\ldots=0.$$

Hence, by Art. 73, the coefficient of α^2 in the point-equation

$$=(\beta_1^2-r^2)(\gamma_1^2-r^2)-(\beta_1\gamma_1+r^2\cos A)^2$$
$$=r^2[r^2\sin^2 A-(\beta_1^2+\gamma_1^2+2\beta_1\gamma_1\cos A)],$$

and the coefficient of $2\beta\gamma$

$$=(\gamma_1\alpha_1+r^2\cos B)(\alpha_1\beta_1+r^2\cos C)-(\alpha_1^2-r^2)(\beta_1\gamma_1+r^2\cos A)$$
$$=r^2[r^2\sin B\sin C+\beta_1\gamma_1+\gamma_1\alpha_1\cos C+\alpha_1\beta_1\cos B-\alpha_1^2\cos A].$$

Similarly for the other terms.

Hence the point-equation as stated.

Cor. If p, q, r are the perpendiculars from the angular points of the triangle of reference upon any tangent to the circle whose centre is $(\alpha_1, \beta_1, \gamma_1)$ and whose radius is ρ, then

$$pa\alpha_1+qb\beta_1+rc\gamma_1=\rho(a\alpha_1+b\beta_1+c\gamma_1),$$

or, in areal coordinates,

$$px_1+qy_1+rz_1=\rho(x_1+y_1+z_1).$$

Ex. 4. *Shew that the circles described on the three diagonals of a complete quadrilateral as diameter are coaxal.*

Taking the figure and notation of Art. 65, we have $m\beta-n\gamma=0$ and $m\beta+n\gamma=0$ as the equations to AO_2 and AO_1.

Hence $$\frac{BO_2\sin B}{O_2C\sin C}=\frac{m}{n},\ \text{ i.e. } \frac{BO_2}{O_2C}=\frac{m}{\sin B}\div\frac{n}{\sin C}=-\frac{BO_1}{O_1C}.$$

Hence, by a well-known geometrical proposition, the circle on O_2O_1 as diameter is the locus of a point K such that

$$\frac{BK}{KC}=\frac{m}{\sin B}\div\frac{n}{\sin C},$$

i.e. $$\frac{m^2\sin^2 C}{n^2\sin^2 B}=\frac{BK^2}{KC^2}=\frac{(\gamma^2+\alpha^2+2\alpha\gamma\cos B)\div\sin^2 B}{(\alpha^2+\beta^2+2\alpha\beta\cos C)\div\sin^2 C},$$

i.e. the equation of the circle on O_1O_2 as diameter is

$$\frac{\gamma^2+\alpha^2+2\alpha\gamma\cos B}{m^2}=\frac{\alpha^2+\beta^2+2\alpha\beta\cos C}{n^2}.$$

So the equations of the circles on PR and QS similarly are

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{l^2}=\frac{\gamma^2+\alpha^2+2\alpha\gamma\cos B}{m^2}$$

and

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{l^2}=\frac{\alpha^2+\beta^2+2\alpha\beta\cos C}{n^2}.$$

These three circles clearly meet in the same two points; for, at the intersection K of the first two, we have

$$\frac{BK^2}{KC^2}=\frac{m^2}{\sin^2 B}\div\frac{n^2}{\sin^2 C}\text{ and }\frac{CK^2}{KA^2}=\frac{n^2}{\sin^2 C}\div\frac{l^2}{\sin^2 A},$$

and hence

$$\frac{BK^2}{KA^2}=\frac{m^2}{\sin^2 B}\div\frac{l^2}{\sin^2 A},$$

i.e. K also lies on the third circle.

It will be found in Art. 99, Ex. 6, that this proposition is a particular case of a more general theorem.

Ex. 5. *Prove Feuerbach's Theorem that the nine-point circle of any triangle touches both the inscribed circle and the three escribed circles.*

If $S=0$ be the equation to the circumscribing circle and $L=0$ the equation to the line at infinity, then, by Art. 87, the equation of the nine-point circle is

$$S=\tfrac{1}{2}L\left[\alpha\cos A+\beta\cos B+\gamma\cos C\right],$$

and that of the inscribed circle is

$$S=L\left[\frac{(s-a)^2}{bc}\alpha+\frac{(s-b)^2}{ca}\beta+\frac{(s-c)^2}{ab}\gamma\right].$$

Hence, by subtraction, their radical axis is

$$a\alpha\left[bc\cos A-2(s-a)^2\right]+\ldots+\ldots=0,$$

i.e.

$$a\alpha\left[b^2+c^2-a^2-(b+c-a)^2\right]+\ldots+\ldots=0,$$

i.e.

$$\frac{a\alpha}{b-c}+\frac{b\beta}{c-a}+\frac{c\gamma}{a-b}=0.$$

This touches the in-circle

$$\sqrt{a(s-a)\alpha}+\sqrt{b(s-b)\beta}+\sqrt{c(s-c)\gamma}=0,$$

if (by Art. 87)

$$a(s-a)\frac{bc}{(c-a)(a-b)}+\ldots+\ldots=0,$$

i.e. if

$$(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)=0,$$

which is true.

Since the radical axis of the two circles touches one of them, it is clear that the two circles must touch.

Similarly the radical axis of the nine-point circle and the escribed circle opposite A is

$$a\alpha(bc\cos A-2s^2)$$
$$+b\beta\left[ca\cos B-2(s-c)^2\right]+c\gamma\left[ab\cos C-2(s-b)^2\right]=0,$$

i.e., on reduction, $$\frac{a\alpha}{b-c}+\frac{b\beta}{c+a}-\frac{c\gamma}{a+b}=0.$$

By Art. 87 this touches the escribed circle

$$\sqrt{-as\alpha}+\sqrt{b(s-c)\beta}+\sqrt{c(s-b)\gamma}=0$$

if $$\frac{as\,.\,bc}{(c+a)(a+b)}-\frac{b(s-c)ca}{(b-c)(a+b)}+\frac{c(s-b)ab}{(b-c)(c+a)}=0,$$

which is true. Similarly for the other two escribed circles.

Ex. 6. *Shew that the equation of the straight lines joining the point A of the triangle of reference to the circular points at infinity is $\beta^2+\gamma^2+2\beta\gamma\cos A=0$, and that with AB and AC they give a pencil of cross-ratio equal to $\cos 2A+\sqrt{-1}\sin 2A$. Prove also that they make the same angle $\tan^{-1}\sqrt{-1}$ with every straight line in the plane.*

The circular points at infinity are the intersections of $a\alpha+b\beta+c\gamma=0$ with any circle such as the circumcircle $a\beta\gamma+b\gamma\alpha+c\alpha\beta=0$. Eliminating α between them, we see that the required lines are

$$-a^2\beta\gamma+(b\gamma+c\beta)(b\beta+c\gamma)=0,$$

i.e. $$\beta^2+\gamma^2+2\beta\gamma\cos A=0.$$

These are the straight lines $\frac{\beta}{\gamma}=-(\cos A\pm i\sin A)=-e^{\pm Ai}$, and by Art. 61 the cross-ratio required

$$=e^{Ai}\div e^{-Ai}=e^{2Ai}=\cos 2A+i\sin 2A.$$

Similarly, in Cartesian coordinates the lines from the origin to the circular points are $x^2+y^2=0$, *i.e.*

$$y=\pm x\sqrt{-1}.$$

The line $y=xi$ makes with any line $y=mx+c$ an angle θ, such that

$$\theta=\tan^{-1}\frac{m-i}{1+mi}=\tan^{-1}(-i).$$

So the other makes with it an angle $\tan^{-1}(i)$.

Cor. By solving the equations

$$\beta^2+\gamma^2+2\beta\gamma\cos A=0 \text{ and } a\alpha+b\beta+c\gamma=0,$$

we can easily shew that the coordinates of the circular points at infinity may be taken to be

$$(\cos B\pm\sqrt{-1}\sin B,\ \cos A\mp\sqrt{-1}\sin A,\ -1).$$

Ex. 7. *To obtain the radical axis of two circles given by the general equations*

$$S_1=a_1\alpha^2+b_1\beta^2+c_1\gamma^2+2f_1\beta\gamma+2g_1\gamma\alpha+2h_1\alpha\beta=0,$$

and $$S_2=a_2\alpha^2+b_2\beta^2+c_2\gamma^2+2f_2\beta\gamma+2g_2\gamma\alpha+2h_2\alpha\beta=0.$$

As in Art. 85, we see that the general equation $S_1=0$ has to be divided by a quantity λ_1 to make it become of the form

$$S=L\,.\,L_1,$$

where S is the equation to the circum-circle and L the equation to the line at infinity, so that

$$\frac{S_1}{\lambda_1}=S-L\,.\,L_1.$$

Similarly $$\frac{S_2}{\lambda_2}=S-L\,.\,L_2.$$

On subtraction $$\frac{S_1}{\lambda_1}-\frac{S_2}{\lambda_2}=-L\,(L_1-L_2).$$

But $L_1-L_2=0$ is the radical axis of the two circles. Hence $\frac{S_1}{\lambda_1}-\frac{S_2}{\lambda_2}=0$ gives the product of the radical axis and the line at infinity.

Also, as in Art. 85, λ_1 and λ_2 are given by

$$b_1c_0^2+c_1b_0^2-2f_1b_0c_0=c_1a_0^2+a_1c_0^2-2g_1c_0a_0=a_1b_0^2+b_1a_0^2-2h_1a_0b_0$$
$$=a_0b_0c_0\lambda_1,$$

with similar expressions for λ_2.

EXAMPLES.

1. Shew that the equations to the circles through the points B and C of the triangle of reference and (1) the ortho-centre, (2) the circum-centre, and (3) the in-centre are

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=l\alpha\,(a\alpha+b\beta+c\gamma),$$

where (1) $l=2\cos A$, (2) $l=\frac{1}{2\cos A}$, and (3) $l=1$.

2. Find the equation of a circle which has for diameter the perpendicular drawn from the angular point A of the triangle of reference to the side BC. Find also the radical axis of this circle and the circum-circle.

3. AD, BE are the perpendiculars from A, B upon the opposite sides of the triangle of reference; shew that the equations to the circles (1) on DE, (2) on the median through A, as diameters are

(1) $a\beta\gamma+b\gamma\alpha+c\alpha\beta=(a\alpha+b\beta+c\gamma)\,[(\alpha\cos A+\beta\cos B)\sin^2 C+\gamma\cos^2 C]$

and (2) $a\beta\gamma+b\gamma\alpha+c\alpha\beta=\frac{1}{2}\,(a\alpha+b\beta+c\gamma)\,(\beta\cos B+\gamma\cos C)$.

4. If from any point P perpendiculars be drawn to the sides of a triangle, and the area of the triangle formed by joining their feet is constant, the locus of P is a circle concentric with the circum-circle of the triangle.

[By Art. 83, the equation obtained is $S=(a\alpha+b\beta+c\gamma)^2\times$ a constant, which is a concentric circle.]

5. Shew that the equation $\beta^2+\gamma^2+2\beta\gamma\cos A=k^2$ represents a circle, and that the radical axis of it and the circum-circle of the triangle of reference is

$$bc\sin^2 A\,(c\beta+b\gamma)=k^2\,(a\alpha+b\beta+c\gamma),$$

and is thus parallel to the tangent at A to the circum-circle.

6. A triangle is inscribed in a circle; shew that the intersections of each side with the tangent at the opposite angular points lie on a straight line.

7. O is a point on the circum-circle of a triangle ABC, and OA, OB, OC meet the opposite sides in a, b, c respectively. Shew that the straight line on which lie the points of intersection of ab and AB, bc and BC, ca and CA passes through a point which is the same for all positions of O.

8. If the sides of the pedal triangle of the triangle ABC are produced to meet the opposite sides in D, E, F, then the straight line DEF is the radical axis of the circum-circle and self-conjugate circle of the triangle ABC.

9. Shew that the equation of a circle, which is of radius ρ and which is concentric with the circum-circle of the triangle of reference, is

$$abc\,(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=(R^2-\rho^2)\,(a\alpha+b\beta+c\gamma)^2.$$

10. Shew that the radius of the circle, whose equation is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=k\alpha\,(a\alpha+b\beta+c\gamma),$$

is $R\sqrt{1-2k\cos A+k^2}$, where R is the radius of the circum-circle of the triangle of reference.

11. If the equation $\alpha\gamma=k\beta^2$ represents a circle, shew that $k=1$ and give the geometrical interpretation.

12. Shew that in areal coordinates the equation

$$a^2(y+z-x)^{-1}+b^2(z+x-y)^{-1}+c^2(x+y-z)^{-1}=0$$

represents a circle, and state its position relative to the fundamental triangle.

13. Find the trilinear equations to the fourth common tangent to the in-circle and the three escribed circles of the triangle of reference, and shew that they are parallel to the tangents to the circum-circle at the vertices of the triangle.

14. Exhibit the equation of the nine-point circle of the triangle of reference in the form

$$\begin{vmatrix} 0, & \gamma, & \beta, & \sin A \\ \gamma, & 0, & \alpha, & \sin B \\ \beta, & \alpha, & 0, & \sin C \\ \cos A, & \cos B, & \cos C, & 0 \end{vmatrix}=0.$$

What geometrical property of the circle does this equation express?

15. Shew that the equation to the pedal line of any point $(\alpha', \beta', \gamma')$ on the circum-circle of the triangle of reference is

$$\alpha\alpha'\,\frac{(\beta'+\gamma'\cos A)\,(\gamma'+\beta'\cos A)}{\sin A}+\ldots+\ldots=0.$$

89. *Foci of the conic given by the general equation.*

To find the foci proceed as in Part I, Art. 393. By Art. 79 write down the equation to the tangents drawn from any point $(\alpha', \beta', \gamma')$ and introduce the conditions found in Art. 85 that they shall satisfy the conditions for being a circle. We shall find that α', β', γ' are given by the equations

$$(b_0^2c + c_0^2b - 2b_0c_0f)\,\phi(\alpha, \beta, \gamma) - [b_0(g\alpha + f\beta + c\gamma) \\ - c_0(h\alpha + b\beta + f\gamma)]^2$$

= two similar expressions, *i.e.* by

$$4(b_0^2c + c_0^2b - 2b_0c_0f)\,\phi(\alpha, \beta, \gamma) - \left[b_0\frac{d\phi}{d\gamma} - c_0\frac{d\phi}{d\beta}\right]^2$$

$$= 4(c_0^2a + a_0^2c - 2c_0a_0g)\,\phi(\alpha, \beta, \gamma) - \left[c_0\frac{d\phi}{d\alpha} - a_0\frac{d\phi}{d\gamma}\right]^2$$

$$= 4(a_0^2b + b_0^2a - 2a_0b_0h)\,\phi(\alpha, \beta, \gamma) - \left[a_0\frac{d\phi}{d\beta} - b_0\frac{d\phi}{d\alpha}\right]^2.$$

When the foci have been found, the directrices, which are their polars, can be at once written down.

If we eliminate $\phi(\alpha, \beta, \gamma)$ from these equations, we shall obtain an equation of the form

$$K\left[b_0\frac{d\phi}{d\gamma} - c_0\frac{d\phi}{d\beta}\right]^2 + L\left[c_0\frac{d\phi}{d\alpha} - a_0\frac{d\phi}{d\gamma}\right]^2 \\ + M\left[a_0\frac{d\phi}{d\beta} - b_0\frac{d\phi}{d\alpha}\right]^2 = 0 \;\ldots(1),$$

where K, L, M are constants.

This is a conic which passes through the foci. It also passes through the centre. For each term of equation (1) is satisfied at the centre as is seen from the equations giving the centre (Art. 78). Since it passes through the four foci and the centre, it can only represent the axes of the conic. Hence the **axes** are obtained by eliminating $\phi(\alpha, \beta, \gamma)$ from the above equations which give the foci.

90. *Conics circumscribing the triangle of reference.*

The general equation to a conic is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

If the conic passes through the vertex A, the equation must be satisfied by $\beta=0$, $\gamma=0$, and $\alpha=$ constant. Hence $a=0$. So, if it passes through the points B and C, we have $b=0$ and $c=0$.

The equation to a circum-conic is thus

$$2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0,$$

or, as we shall write it,

$$L\beta\gamma+M\gamma\alpha+N\alpha\beta=0.$$

As in Art. 72, it is easily seen that the straight line $l\alpha+m\beta+n\gamma=0$ touches this conic if

$$L^2l^2+M^2m^2+N^2n^2-2MNmn-2NLnl-2LMlm=0,$$

i.e. if $$\sqrt{Ll}+\sqrt{Mm}+\sqrt{Nn}=0.$$

Ex. *The equation to the straight line joining the points* $(\alpha', \beta', \gamma')$ *and* $(\alpha'', \beta'', \gamma'')$ *on the circum-conic is* $\frac{L\alpha}{\alpha'\alpha''}+\frac{M\beta}{\beta'\beta''}+\frac{N\gamma}{\gamma'\gamma''}=0$, *and the tangent at* $(\alpha', \beta', \gamma')$ *is* $\frac{L\alpha}{\alpha'^2}+\frac{M\beta}{\beta'^2}+\frac{N\gamma}{\gamma'^2}=0$.

Since the points lie on the curve, we have

$$L\beta'\gamma'+M\gamma'\alpha'+N\alpha'\beta'=0 \text{ and } L\beta''\gamma''+M\gamma''\alpha''+N\alpha''\beta''=0.$$

Hence

$$\frac{L}{\alpha'\alpha''(\beta'\gamma''-\beta''\gamma')}=\frac{M}{\beta'\beta''(\gamma'\alpha''-\gamma''\alpha')}=\frac{N}{\gamma'\gamma''(\alpha'\beta''-\alpha''\beta')}.$$

But the equation to the straight line joining the two given points is, by Art. 51,

$$\alpha(\beta'\gamma''-\beta''\gamma')+\beta(\gamma'\alpha''-\gamma''\alpha')+\gamma(\alpha'\beta''-\alpha''\beta')=0,$$

i.e. $$\frac{L\alpha}{\alpha'\alpha''}+\frac{M\beta}{\beta'\beta''}+\frac{N\gamma}{\gamma'\gamma''}=0.$$

On putting $\alpha''=\alpha'$, $\beta''=\beta'$, $\gamma''=\gamma'$, we have the equation to the tangent at $(\alpha', \beta', \gamma')$.

91. *Conics inscribed in the triangle of reference.*

The general equation to a conic is

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0.$$

This meets BC, *i.e.* $\alpha=0$, where $b\beta^2+c\gamma^2+2f\beta\gamma=0$, and therefore touches it if

$$f^2=bc, \text{ i.e. if } f=\pm\sqrt{bc}.$$

So the conic touches the other two sides if

$$g=\pm\sqrt{ca}, \text{ and } h=\pm\sqrt{ab}.$$

Putting $a = L^2$, $b = M^2$, and $c = N^2$, we have as the equation to the inscribed conic

$$L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 \pm 2MN\beta\gamma \pm 2NL\gamma\alpha \pm 2LM\alpha\beta = 0.$$

These ambiguities in sign are not however all independent; for, if they were replaced by all positive signs, or if they were replaced by two negative and one positive signs, we should in either case get a perfect square, and the locus given would be two coincident straight lines.

[Two coincident straight lines clearly satisfy the condition of passing through two coincident points on the side of the triangle.]

Hence the ambiguities may only be replaced by (1) three negative signs, or (2) by one negative and two positive signs.

The equation may be written in the form

$$\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0.$$

Taking the equation

$$\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0,$$

i.e. $L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0,$

we see, from Art. 72, that the straight line

$$l\alpha + m\beta + n\gamma = 0$$

touches it if $\qquad Lmn + Mnl + Nlm = 0,$

i.e. we see that the line (l, m, n) touches the in-conic

$$\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0$$

if the point (l, m, n) lies on the corresponding circum-conic

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0.$$

Also similarly, from Art. 90, we see that the line (l, m, n) touches the circum-conic

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0$$

if the point (l, m, n) lies on the corresponding in-conic

$$\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0.$$

Ex. *The equation to the chord joining the points* $(\alpha', \beta', \gamma')$ *and* $(\alpha'', \beta'', \gamma'')$ *on the in-conic is*

$$\alpha\sqrt{L}\left[\sqrt{\beta'\gamma''} + \sqrt{\gamma'\beta''}\right] + \beta\sqrt{M}\left[\sqrt{\gamma'\alpha''} + \sqrt{\alpha'\gamma''}\right] \\ + \gamma\sqrt{N}\left[\sqrt{\alpha'\beta''} + \sqrt{\beta'\alpha''}\right] = 0,$$

and the tangent at $(\alpha', \beta', \gamma')$ *is* $\alpha\sqrt{\frac{L}{\alpha'}}+\beta\sqrt{\frac{M}{\beta'}}+\gamma\sqrt{\frac{N}{\gamma'}}=0.$

Since the points lie on the in-conic, we have

$$\sqrt{L\alpha'}+\sqrt{M\beta'}+\sqrt{N\gamma'}=0 \text{ and } \sqrt{L\alpha''}+\sqrt{M\beta''}+\sqrt{N\gamma''}=0.$$

Hence

$$\frac{\sqrt{L}}{\sqrt{\beta'\gamma''}-\sqrt{\beta''\gamma'}}=\frac{\sqrt{M}}{\sqrt{\gamma'\alpha''}-\sqrt{\gamma''\alpha'}}=\frac{\sqrt{N}}{\sqrt{\alpha'\beta''}-\sqrt{\alpha''\beta'}}.$$

$$\therefore \frac{\beta'\gamma''-\beta''\gamma'}{\sqrt{L}\left[\sqrt{\beta'\gamma''}+\sqrt{\beta''\gamma'}\right]}=\frac{\gamma'\alpha''-\gamma''\alpha'}{\sqrt{M}\left[\sqrt{\gamma'\alpha''}+\sqrt{\gamma''\alpha'}\right]}$$

$$=\frac{\alpha'\beta''-\alpha''\beta'}{\sqrt{N}\left[\sqrt{\alpha'\beta''}+\sqrt{\alpha''\beta'}\right]}.$$

Hence the equation to the required chord (Art. 51) is

$$\alpha\sqrt{L}\left[\sqrt{\beta'\gamma''}+\sqrt{\beta''\gamma'}\right]+\beta\sqrt{M}\left[\sqrt{\gamma'\alpha''}+\sqrt{\gamma''\alpha'}\right]$$

$$\dots+\gamma\sqrt{N}\left[\sqrt{\alpha'\beta''}+\sqrt{\alpha''\beta'}\right]=0.$$

On putting $\alpha''=\alpha'$, $\beta''=\beta'$, $\gamma''=\gamma'$ the equation to the tangent at $(\alpha', \beta', \gamma')$ is then

$$\alpha\sqrt{L\beta'\gamma'}+\beta\sqrt{M\gamma'\alpha'}+\gamma\sqrt{N\alpha'\beta'}=0,$$

i.e.

$$\alpha\sqrt{\frac{L}{\alpha'}}+\beta\sqrt{\frac{M}{\beta'}}+\gamma\sqrt{\frac{N}{\gamma'}}=0.$$

92. Ex. 1. *A conic circumscribes a given triangle and one of its asymptotes passes through a fixed point; shew that the other asymptote touches a fixed conic inscribed in the triangle.*

Let $\qquad l_1\alpha+m_1\beta+n_1\gamma=0$ and $l_2\alpha+m_2\beta+n_2\gamma=0$

be the asymptotes. Then the equation to the conic must be

$$(l_1\alpha+m_1\beta+n_1\gamma)(l_2\alpha+m_2\beta+n_2\gamma)=k(a\alpha+b\beta+c\gamma)^2,$$

since the conic touches the two asymptotes where they meet the line at infinity.

Since the conic circumscribes the triangle of reference, its equation has no terms containing α^2, β^2, or γ^2,

$$\therefore l_1l_2-ka^2=0,\ m_1m_2-kb^2=0, \text{ and } n_1n_2-kc^2=0.$$

Since one asymptote passes through a fixed point (f, g, h),

$$\therefore l_1f+m_1g+n_1h=0.$$

$$\therefore \frac{a^2f}{l_2}+\frac{b^2g}{m_2}+\frac{c^2h}{n_2}=0.$$

$$\therefore a^2fm_2n_2+b^2gn_2l_2+c^2hl_2m_2=0.$$

Hence the other asymptote touches the inscribed conic

$$\sqrt{a^2f\alpha}+\sqrt{b^2g\beta}+\sqrt{c^2h\gamma}=0. \qquad \text{(Art. 91.)}$$

Ex. 2. *Shew that the locus of the foci of conics which touch four given straight lines is in general a curve of the third degree.*

Take three of the tangents as the triangle of reference, and let the fourth tangent be

$$l\alpha+m\beta+n\gamma=0.$$

Let the foci be $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$. Then, since the product of the perpendiculars from the foci upon any tangent is constant,

$$\therefore \alpha'\alpha''=\beta'\beta''=\gamma'\gamma''=\frac{(l\alpha'+m\beta'+n\gamma')(l\alpha''+m\beta''+n\gamma'')}{L^2}=k^2,$$

where $\quad L^2=l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C.$

$$\therefore k^2L^2=(l\alpha'+m\beta'+n\gamma')\left(\frac{k^2l}{\alpha'}+\frac{k^2m}{\beta'}+\frac{k^2n}{\gamma'}\right).$$

Hence the locus of $(\alpha', \beta', \gamma')$ is the cubic curve

$$(l\alpha+m\beta+n\gamma)(l\beta\gamma+m\gamma\alpha+n\alpha\beta)$$
$$=L^2\alpha\beta\gamma=\alpha\beta\gamma(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C).$$

This curve passes through the six intersections of the four given straight lines.

If $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$, *i.e.* if the fourth tangent is the line at infinity and hence the conics are parabolas, this locus reduces to

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$

i.e. the circum-circle of the triangle formed by the other three tangents.

If the four tangents form a parallelogram then, as in Part I, page 376, Ex. 11, the locus is a rectangular hyperbola.

Ex. 3. *If the inscribed conic* $\sqrt{L\alpha}+\sqrt{M\beta}+\sqrt{N\gamma}=0$ *be a parabola, shew that its focus is the point* $\left(\frac{a^2}{L}, \frac{b^2}{M}, \frac{c^2}{N}\right)$, *and that the equation to its directrix is* $L\alpha\cot A+M\beta\cot B+N\gamma\cot C=0$.

Shew also that the directrices of all such parabolas pass through the orthocentre of the triangle of reference and that the locus of their foci is the circum-circle.

The tangential equation to the conic is

$$Lmn+Mnl+Nlm=0,$$

and hence the line at infinity is a tangent, and therefore the conic is a parabola, if

$$Lbc+Mca+Nab=0 \text{ ....}(1).$$

For any conic inscribed in the triangle of reference, we have

$$\alpha_1\alpha_2=\beta_1\beta_2=\gamma_1\gamma_2=(\text{semi-minor axis})^2 \text{ ....}(2),$$

where $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are the foci.

If the conic is a parabola, one focus is the point of contact of the line at infinity and is thus given by

$$\frac{L(L\alpha_2 - M\beta_2 - N\gamma_2)}{a} = \frac{M(-L\alpha_2 + M\beta_2 - N\gamma_2)}{b} = \frac{N(-L\alpha_2 - M\beta_2 + N\gamma_2)}{c}.$$

Hence $\alpha_2 : \beta_2 : \gamma_2 :: Mc + Nb : Na + Lc : Lb + Ma :: \frac{L}{a^2} : \frac{M}{b^2} : \frac{N}{c^2}$, by equation (1).

Hence (2) gives $\qquad \alpha_1 : \beta_1 : \gamma_1 :: \frac{a^2}{L} : \frac{b^2}{M} : \frac{c^2}{N}$ ....(3).

The directrix is the polar of the focus, and hence is

$$L\alpha(L\alpha_1 - M\beta_1 - N\gamma_1) + \ldots + \ldots = 0,$$

i.e. $$L\alpha(a^2 - b^2 - c^2) + \ldots + \ldots = 0,$$

i.e. $$L\alpha \cot A + M\beta \cot B + N\gamma \cot C = 0.$$

From (1) and (3) we have $a\beta_1\gamma_1 + b\gamma_1\alpha_1 + c\alpha_1\beta_1 = 0$, so that the focus lies on the circum-circle.

Also the directrix clearly passes through the orthocentre, *i.e.* the point ($\cos B \cos C$, $\cos C \cos A$, $\cos A \cos B$), since (1) is true.

Ex. 4. *Shew that the locus of the centres of rectangular hyperbolas inscribed in the triangle of reference is the self-conjugate circle*

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0.$$

An inscribed rectangular hyperbola is

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0 \quad \ldots\ldots(1),$$

where $$l^2 + m^2 + n^2 + 2mn\cos A + 2nl\cos B + 2lm\cos C = 0 \quad \ldots\ldots(2).$$

The centre of (1) is given by

$$\frac{l(l\bar{\alpha} - m\bar{\beta} - n\bar{\gamma})}{a} = \frac{m(m\bar{\beta} - n\bar{\gamma} - l\bar{\alpha})}{b} = \frac{n(n\bar{\gamma} - l\bar{\alpha} - m\bar{\beta})}{c}.$$

$$\therefore \frac{2l\bar{\alpha}}{\frac{b}{m} + \frac{c}{n}} = \frac{2m\bar{\beta}}{\frac{c}{n} + \frac{a}{l}} = \frac{2n\bar{\gamma}}{\frac{a}{l} + \frac{b}{m}}.$$

$$\therefore \frac{bn + cm}{\bar{\alpha}} = \frac{cl + an}{\bar{\beta}} = \frac{am + bl}{\bar{\gamma}}.$$

$$\therefore \frac{lbc}{b\bar{\beta} + c\bar{\gamma} - a\bar{\alpha}} = \frac{mca}{c\bar{\gamma} + a\bar{\alpha} - b\bar{\beta}} = \frac{nab}{a\bar{\alpha} + b\bar{\beta} - c\bar{\gamma}}.$$

Substituting in (2), we have

$$a^2(b\bar{\beta} + c\bar{\gamma} - a\bar{\alpha})^2 + \ldots + \ldots$$
$$+ 2\cos A \,.\, bc(c\bar{\gamma} + a\bar{\alpha} - b\bar{\beta})(a\bar{\alpha} + b\bar{\beta} - c\bar{\gamma}) + \ldots + \ldots = 0.$$

Hence, on reduction, the required locus is

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C = 0,$$

which is the self-polar circle. (Art. 87.)

EXAMPLES.

1. Shew that the equation to the conic circumscribing the triangle of reference, such that the tangent at each of its angular points is parallel to the opposite side, is $bc\,\beta\gamma + ca\,\gamma\alpha + ab\,\alpha\beta = 0$.

2. Deduce the equation to the circum-circle of the triangle of reference from the fact that the tangent at A makes with AC an angle equal to B, and similarly for the other angular points.

3. Two conics circumscribe the triangle ABC; through A is drawn any straight line to meet them in P and Q; shew that the tangents at P and Q divide BC in constant cross-ratio.

4. Shew that the equations

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} = 0 \text{ and } \sqrt{\frac{\beta}{m} + \frac{\gamma}{n}} + \sqrt{\frac{\gamma}{n} + \frac{\alpha}{l}} + \sqrt{\frac{\alpha}{l} + \frac{\beta}{m}} = 0$$

represent the same conic, and give the geometrical interpretation.

5. A conic circumscribes the triangle of reference and touches the line $l\alpha + m\beta + n\gamma = 0$; shew that the locus of the pole of the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$ with respect to the conic is

$$\sqrt{l\alpha(\mu\beta + \nu\gamma - \lambda\alpha)} + \sqrt{m\beta(\nu\gamma + \lambda\alpha - \mu\beta)} + \sqrt{n\gamma(\lambda\alpha + \mu\beta - \nu\gamma)} = 0.$$

Deduce the locus of its centre.

6. If the circum-conic $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ is a parabola, shew that the equation to its directrix is

$$bc(m^2 + n^2 - 2mn\cos A)\,\alpha + ca(n^2 + l^2 - 2nl\cos B)\,\beta$$
$$+ ab(l^2 + m^2 - 2lm\cos C)\,\gamma = 0.$$

If it touches the circum-circle of the triangle of reference at A, shew that it is one of two parabolas whose axes are at right angles.

7. If the normals to the conic $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ at the vertices of the triangle of reference are concurrent, prove that

$$(m - n\cos A)(n - l\cos B)(l - m\cos C)$$
$$= (n - m\cos A)(l - n\cos B)(m - l\cos C).$$

8. Shew that the equation to the conic which has contact of the second order with the conic $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$ at the vertex A of the triangle of reference, and which passes through the vertex B and the in-centre, is

$$(m + n)(l\beta\gamma + m\gamma\alpha + n\alpha\beta) = (l + m + n)(n\beta + m\gamma)\,\gamma.$$

9. A triangle ABC is inscribed in a conic; the polar of a point A_1, lying on BC, meets CA, AB in B_1 and C_1. Shew that AA_1, BB_1 and CC_1 meet in a point.

10. Shew that the coordinates of any point on the circum-conic $L\beta\gamma+M\gamma\alpha+N\alpha\beta=0$ can be written in the form

$$(L\operatorname{cosec}^2\theta,\ M\sec^2\theta,\ -N),$$

and that the tangent at that point is

$$\frac{\alpha}{L}\sin^4\theta+\frac{\beta}{M}\cos^4\theta+\frac{\gamma}{N}=0.$$

So for the in-conic $\sqrt{L\alpha}+\sqrt{M\beta}+\sqrt{N\gamma}=0$, a point is

$$\left(\frac{\cos^4\theta}{L},\ \frac{\sin^4\theta}{M},\ \frac{1}{N}\right),$$

and the tangent is $$\frac{L\alpha}{\cos^2\theta}+\frac{M\beta}{\sin^2\theta}-N\gamma=0.$$

11. Whatever be the value of l, m and n, shew that the equation

$$(m-n)\sqrt{\beta+l\gamma}+(n-l)\sqrt{\beta+m\gamma}+(l-m)\sqrt{\beta+n\gamma}=0$$

represents only the line $\gamma=0$.

Interpret the equation $\lambda\sqrt{L}+\mu\sqrt{M}+\nu\sqrt{N}=0$, where L, M, N are (1) general, (2) concurrent, straight lines.

12. Shew that $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$ and

$$(-lx+my+nz)^{-1}+(lx-my+nz)^{-1}+(lx+my-nz)^{-1}=0$$

represent the same conic, and give the geometrical reason for this.

13. A conic is inscribed in a given triangle and passes through a given point; shew that the locus of its centre is a conic which touches the three lines joining the middle points of the sides of the given triangle.

14. ABC is a triangle and P any point on a fixed straight line. Shew that the envelope of the harmonic conjugate of PA with regard to PB, PC is a conic touching the sides of the triangle ABC and the fixed straight line.

15. Parabolas are inscribed in a given triangle; shew that the line joining the points of contact of any two sides passes through a fixed point.

16. Shew that the equation

$$\sqrt{a(b-c)\alpha}+b\sqrt{-\beta}+c\sqrt{\gamma}=0$$

represents a parabola whose axis is the external bisector of the angle ABC.

17. Shew that the equation

$$\sqrt{a\alpha}+\sqrt{b\beta}+\sqrt{-c\gamma}=0$$

represents a hyperbola whose asymptotes are parallel to the lines $\alpha=0$ and $\beta=0$.

18. A conic touches the sides BC, CA, AB of a triangle in P, Q, R. Shew that AP, BQ and CR are concurrent and that the pencil $P(QARB)$ is harmonic.

If AP, BQ, CR meet the conic again in P', Q', R', shew that the tangents at P', Q', R' meet the sides BC, CA, AB in collinear points.

19. A conic, whose centre is O, touches the sides BC, CA, AB of the triangle ABC in the points D, E, F respectively; DO, EO, FO meet EF, FD, DE in L, M, N respectively and MN, NL, LM intersect BC, CA, AB in X, Y, Z respectively. Shew that X, Y, Z are collinear.

If D', E', F' are the other ends of the diameters through D, E, F, shew that AD', BE', CF' are concurrent.

20. Shew that the triangle formed by the poles of the lines joining the middle points of the sides of a given triangle with regard to any inscribed conic is of constant area.

21. Shew that, in areal coordinates, $px+qy+rz=0$ is an asymptote of the conic

$$\sqrt{Lx}+\sqrt{My}+\sqrt{Nz}=0,$$

if p, q, r are given by

$$Lqr+Mrp+Npq=0,$$

and

$$p(M+N)+q(N+L)+r(L+M)=0.$$

22. A conic is inscribed in a triangle; shew that its foci lie on a cubic curve circumscribing the triangle if the major axis passes through a fixed point, or if the centre lies on a fixed line.

23. A conic is inscribed in the triangle of reference; if its minor axis is of given length 2λ, the locus of its foci is the cubic curve

$$4\Delta^2\alpha\beta\gamma=\lambda^2(a\alpha+b\beta+c\gamma)(a\beta\gamma+b\gamma\alpha+c\alpha\beta),$$

where Δ is the area of the triangle of reference.

If its axis is parallel to the straight line $l\alpha+m\beta+n\gamma=0$, their locus is the cubic

$$\alpha(\beta^2-\gamma^2)(mc-nb)+\beta(\gamma^2-\alpha^2)(na-lc)+\gamma(\alpha^2-\beta^2)(lb-ma)=0.$$

24. Find the equation, in areal coordinates, of the conic which touches the sides of the triangle of reference at their middle points, and verify that the centre of the conic is the centroid of the triangle.

Shew that the equation to the axes of the conic may be written in the form

$$(b^2-c^2)(y-z)^2+(c^2-a^2)(z-x)^2+(a^2-b^2)(x-y)^2=0.$$

93. *Conics with respect to which the triangle of reference is self-conjugate.*

The polar of the point A viz. $(\alpha', 0, 0)$ with respect to the conic

$$\phi(\alpha, \beta, \gamma) \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta$$

is

$$\alpha'[a\alpha + h\beta + g\gamma] = 0. \qquad \text{(Art. 75)}$$

This is the opposite side BC, *i.e.* $\alpha = 0$, if $g = h = 0$. So the polar of the point B is the opposite side CA if $h = f = 0$.

Since the polars of the points A and B pass through C, therefore, by Part I, Art. 375, the polar of C passes through A and B.

Hence, if $f = g = h = 0$, the triangle ABC is self-conjugate with respect to the conic.

The equation to such a conic is usually written in the form

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0.$$

A conic with respect to which a triangle is self-conjugate, or self-polar, is often called a self-conjugate conic with respect to the triangle, or, for brevity, "a self-conjugate conic," it being understood, when this latter expression is used, that the words "with regard to the triangle of reference" are implied.

94. *Condition that $l\alpha + m\beta + n\gamma = 0$ may be a tangent to $L\alpha^2 + M\beta^2 + N\gamma^2 = 0$.*

The tangent at any point $(\alpha', \beta', \gamma')$ of the conic is

$$L\alpha\alpha' + M\beta\beta' + N\gamma\gamma' = 0.$$

If this be the given straight line, then

$$\frac{L\alpha'}{l} = \frac{M\beta'}{m} = \frac{N\gamma'}{n},$$

so that

$$\alpha' : \beta' : \gamma' :: \frac{l}{L} : \frac{m}{M} : \frac{n}{N}.$$

Since $(\alpha', \beta', \gamma')$ lies on the conic,

$$\therefore \frac{l^2}{L} + \frac{m^2}{M} + \frac{n^2}{N} = 0.$$

This is the required condition of tangency.

It is clear that if $l\alpha + m\beta + n\gamma = 0$ is a tangent, so also are the four lines $\pm l\alpha \pm m\beta \pm n\gamma = 0$.

Cor. The conic is a parabola if it touches the line at infinity, $a_0\alpha+b_0\beta+c_0\gamma=0$, the condition for which is

$$\frac{a_0^2}{L}+\frac{b_0^2}{M}+\frac{c_0^2}{N}=0.$$

This parabola will also touch the three straight lines

$$a_0\alpha+b_0\beta-c_0\gamma=0,\ a_0\alpha-b_0\beta+c_0\gamma=0 \text{ and } -a_0\alpha+b_0\beta+c_0\gamma=0.$$

But these are the three straight lines joining the middle points of the sides of the triangle of reference.

Hence *If a parabola be self-conjugate with respect to a triangle ABC, it is inscribed in the triangle formed by joining the middle points of the sides of the triangle ABC.*

95. Ex. *If a triangle is self-conjugate with respect to a rectangular hyperbola, its circum-circle passes through the centre of the hyperbola. If it is inscribed in the rectangular hyperbola, then its nine-point circle passes through the centre.*

In the first case the conic is

$$L\alpha^2+M\beta^2+N\gamma^2=0,$$

where

$$L+M+N=0 \quad\text{..............................(1)}$$

by Art. 80.

The centre is given by $\dfrac{L\bar{\alpha}}{a}=\dfrac{M\bar{\beta}}{b}=\dfrac{N\bar{\gamma}}{c}$.

Hence, by (1),

$$a\bar{\beta}\bar{\gamma}+b\bar{\gamma}\bar{\alpha}+c\bar{\alpha}\bar{\beta}=0,$$

i.e. the centre lies on the circum-circle.

In the second case the conic is

$$L\beta\gamma+M\gamma\alpha+N\alpha\beta=0,$$

where

$$L\cos A+M\cos B+N\cos C=0 \quad\text{..................(2).}$$

The centre is given by

$$\frac{M\bar{\gamma}+N\bar{\beta}}{a}=\frac{N\bar{\alpha}+L\bar{\gamma}}{b}=\frac{L\bar{\beta}+M\bar{\alpha}}{c}.$$

$$\therefore\quad \frac{L\bar{\beta}\bar{\gamma}}{-a\bar{\alpha}+b\bar{\beta}+c\bar{\gamma}}=\frac{M\bar{\gamma}\bar{\alpha}}{a\bar{\alpha}-b\bar{\beta}+c\bar{\gamma}}=\frac{N\bar{\alpha}\bar{\beta}}{a\bar{\alpha}+b\bar{\beta}-c\bar{\gamma}}.$$

Hence (2) gives

$$\bar{\alpha}\,(-a\bar{\alpha}+b\bar{\beta}+c\bar{\gamma})\cos A+\ldots+\ldots=0,$$

i.e.

$$a\bar{\beta}\bar{\gamma}+b\bar{\gamma}\bar{\alpha}+c\bar{\alpha}\bar{\beta}-a\cos A\bar{\alpha}^2-b\cos B\bar{\beta}^2-c\cos C\bar{\gamma}^2=0,$$

i.e. the centre lies on the nine-point circle.

The second part follows at once from the first; for, if a rectangular hyperbola circumscribes a triangle, it is easy to shew that the pedal triangle of the latter is self-conjugate with regard to the hyperbola.

96. *To shew that, in general, any two conics have a common self-conjugate triangle, and to find in what cases it is real and when it is imaginary.*

The intersections of two conics are either (1) all real, (2) all imaginary, or (3) two real and two imaginary, since imaginary solutions always occur in pairs.

(1) If the intersections P, Q, R, S are all real, as in the figure of Art. 64, then the triangle ABC is the self-conjugate triangle. For, if AC meets PS in M and QR in N, then, since the pencil $C\,(PASB)$ is harmonic, therefore $[BSMP]$ and $[BRNQ]$ are harmonic, and hence by Part I, Art. 401, CA is the polar of B. Similarly C is the pole of AB and A the pole of BC.

(2) If the intersections are all imaginary,

let P be $(x_1 + x_2 i,\ y_1 + y_2 i)$,
let Q be $(x_1 - x_2 i,\ y_1 - y_2 i)$,
let R be $(x_3 + x_4 i,\ y_3 + y_4 i)$,
and S be $(x_3 - x_4 i,\ y_3 - y_4 i)$,

the coordinates being Cartesian, and $i = \sqrt{-1}$.

Then the equation of PQ is

$$x_2 (y - y_1) = y_2 (x - x_1) \quad \ldots\ldots\ldots\ldots\ldots(1),$$

and that of RS is

$$x_4 (y - y_3) = y_4 (x - x_3) \quad \ldots\ldots\ldots\ldots\ldots(2).$$

That of PR is $L + Mi = 0$, and that of QS is $L - Mi = 0$, where L and M are both wholly real. That of PS is

$$L' + M'i = 0,$$

and of QR is $L' - M'i = 0$,

where L' and M' are wholly real.

Now PQ and RS intersect in a real point C, PR and QS intersect in a real point A, given by $L = 0$ and $M = 0$, and PS and QR meet in a real point B, given by $L_1 = 0$ and $M_1 = 0$.

In this case the self-conjugate triangle is therefore wholly real; but only two of the common chords are real.

(3) Next, let P, Q be real and R, S imaginary. Let

P be (x_1, y_1); Q, (x_2, y_2); R, $(x_3 + x_4 i,\ y_3 + y_4 i)$;

and S, $(x_3 - x_4 i,\ y_3 - y_4 i)$.

Then PQ is

$$y(x_1 - x_2) - x(y_1 - y_2) - (x_1 y_2 - x_2 y_1) = 0,$$

and RS is $$x_4(y - y_3) = y_4(x - x_3).$$

The equation of PR is

$$L_1 + M_1 i = 0,$$

and that of PS is $$L_1 - M_1 i = 0.$$

The equation of QR is

$$L_1' + M_1' i = 0,$$

and that of QS is $$L_1' - M_1' i = 0,$$

where L_1, M_1, L_1', M_1' are all wholly real. PQ and RS intersect in a real point C. Now, since PR passes through the real point P, it cannot pass through any other real point; for then it would be wholly real, which by supposition it is not. So QS has no real point on it except Q. Hence PR and QS cannot meet in a real point. Hence the point A of the figure of Art. 64 is in this case imaginary. Similarly PS and QR cannot intersect in a real point.

In this case therefore only one vertex of the self-conjugate triangle is real, and the others are imaginary, whilst two of the common chords are real.

If the conics touch at a single point, then two of the points of intersection, P and Q, coincide, and it is easily seen that the triangle ABC of Art. 64 becomes evanescent, the points A and B both coinciding with P and Q.

If the conics have double contact then two of the points, P and Q, coincide and the other two, R and S, also coincide. PQ and RS then give a real point C and the lines PR and QS coincide, as also PS and QR, giving in each case an infinite number of points of intersection, and hence an infinite number of self-conjugate triangles. It we take CPR as the triangle of reference, so that the equations of the conics are $k\gamma^2 - 2\alpha\beta = 0$ and $k'\gamma^2 - 2\alpha\beta = 0$, it is easy to shew that, if K and L be any points on PR such that $[PKRL]$ is harmonic, then CKL is a self-conjugate triangle with respect to the two conics.

Hence in this case there is an infinite number of self-conjugate triangles.

If the conics have three or four-point contact, the self-conjugate triangle reduces to an infinitely small triangle at the point of contact.

Ex. *Find the self-conjugate triangle for the conics whose equations, in any system of coordinates, are*

$$S \equiv a(b-c)yz + b(c-a)zx + c(a-b)xy = 0,$$

and

$$S' \equiv a^2(b-c)yz + b^2(c-a)zx + c^2(a-b)xy = 0.$$

Clearly, $kS + S' = 0$ is a pair of straight lines when $k = -a$ or $-b$ or $-c$.

When $k = -a$, we obtain $bzx - cxy = 0$, giving the two lines $x = 0$ and $bz - cy = 0$, which meet at the point $(0, b, c)$.

So when $k = -b$, or $-c$, we have two straight lines meeting at the points $(a, 0, c)$, and $(a, b, 0)$. These three points are the vertices of the self-conjugate triangle.

The equations to its sides are easily seen to be

$$X \equiv -xbc + yca + zab = 0,$$

$$Y \equiv xbc - yca + zab = 0,$$

and

$$Z \equiv xbc + yca - zab = 0,$$

and we have $S = [(b-c)X^2 + (c-a)Y^2 + (a-b)Z^2] \div 4abc$,

and $S' = [a(b-c)X^2 + b(c-a)Y^2 + c(a-b)Z^2] \div 4abc$.

EXAMPLES.

1. A given triangle is self-conjugate with regard to a conic; shew that the locus of the centre of the conic is a straight line if it touch a fourth given straight line, and is a conic circumscribing the given triangle if it passes through a given point.

2. Shew that all parabolas with a given self-conjugate triangle touch the lines joining the middle points of the sides of the triangle, and that there is one such parabola which also touches the sides of the pedal triangle of the given triangle.

3. A series of conics is described having a common self-conjugate triangle; if one asymptote of each passes through a fixed point, shew that the other will envelope a conic.

4. Shew that one asymptote of the conic $ax^2 + by^2 + cz^2 = 0$ will be parallel to an asymptote of the conic $a'x^2 + b'y^2 + c'z^2 = 0$ if

$$\sqrt{bc' - b'c} + \sqrt{ca' - c'a} + \sqrt{ab' - a'b} = 0,$$

the coordinates being areal.

[One point at infinity is the same for both conics.]

5. Shew that the equation to the director circle of the conic

$$l\alpha^2+m\beta^2+n\gamma^2=0$$

is

$$l(m+n)\alpha^2+m(n+l)\beta^2+n(l+m)\gamma^2+2mn\beta\gamma\cos A$$
$$+2nl\gamma\alpha\cos B+2lm\alpha\beta\cos C=0,$$

and that the radical axis of this circle and the circum-circle of the triangle is

$$l(m+n)\,bc\alpha+m(n+l)\,ca\beta+n(l+m)\,ab\gamma=0.$$

6. If the conic $l\alpha^2+m\beta^2+n\gamma^2=0$ is a parabola, shew that the equation to its directrix is

$$l(m+n)\,bc\alpha+m(n+l)\,ca\beta+n(l+m)\,ab\gamma=0,$$

and that its focus is $\left(\dfrac{m+n}{a},\ \dfrac{n+l}{b},\ \dfrac{l+m}{c}\right)$.

Prove that the directrices of all such parabolas pass through the centre of the circum-circle, and that their foci lie on the nine-point circle.

7. Shew that the coordinates of any point on the conic

$$L\alpha^2+M\beta^2+N\gamma^2=0$$

are

$$\left(\frac{\cos\theta}{\sqrt{L}},\ \frac{\sin\theta}{\sqrt{M}},\ \frac{1}{\sqrt{-N}}\right),$$

and that the tangent at this point is

$$\sqrt{L}\alpha\cos\theta+\sqrt{M}\beta\sin\theta-\sqrt{-N}\gamma=0.$$

8. Shew that any tangent to the conic

$$\frac{\alpha^2}{b+1}+\frac{\beta^2}{a+1}=\frac{\gamma^2}{a+b}$$

is cut by the conics $\alpha^2+\beta^2=\gamma^2$ and $a\alpha^2+b\beta^2=\gamma^2$ in two pairs of points which are harmonically conjugate.

[Any point on the first conic is

$$(\sqrt{b+1}\cos\theta,\ \sqrt{a+1}\sin\theta,\ \sqrt{a+b}).]$$

9. If two of the sides of an inscribed triangle of the conic

$$l\alpha^2+m\beta^2=n\gamma^2$$

touch the conic $\qquad ll'\alpha^2+mm'\beta^2=nn'\gamma^2,$

prove that the envelope of the third side is the conic

$$l\alpha^2(-m'n'+n'l'+l'm')^2+m\beta^2(m'n'-n'l'+l'm')^2=n\gamma^2(m'n'+n'l'-l'm')^2.$$

10. A triangle is self-conjugate with regard to a rectangular hyperbola; shew that the foci of any conic inscribed in the triangle are conjugate points with respect to the hyperbola.

11. Shew that all the conics, which touch the four common tangents to the conics

$$\frac{x^2}{l}+\frac{y^2}{m}+\frac{z^2}{n}=0 \text{ and } \frac{x^2}{l'}+\frac{y^2}{m'}+\frac{z^2}{n'}=0,$$

are given by

$$\frac{x^2}{l+kl'}+\frac{y^2}{m+km'}+\frac{z^2}{n+kn'}=0,$$

where k is any parameter.

12. If a series of conics have a common self-conjugate triangle, then any line through a vertex of the triangle is cut by the system in involution, and the tangents drawn from any vertex to the conics form a pencil in involution.

13. Find the vertices of the self-conjugate triangle in the case of the following conics given in Cartesian coordinates, and express them in a self-conjugate form:

(1) $x^2-y^2-2y+1=0$, and $y^2+4xy=4x$;

(2) $7xy+y^2-4x-y=0$, and $2x^2+xy+5y^2-7y+2=0$;

and (3) $4x^2+y^2+6x-4y=0$, and $4xy+3y^2-8x+6y+1=0$.

14. If the equations of two conics in any system of homogeneous coordinates are

$$y^2+yz-zx+xy=0$$

and

$$x^2+y^2-2z^2-2yz-2zx+4xy=0,$$

find the equations to the sides of the self-conjugate triangle, and express the equations of the conics in self-conjugate forms.

15. If three conics $\phi_1(x, y, z)=0$, $\phi_2(x, y, z)=0$ and $\phi_3(x, y, z)=0$ have a common self-conjugate triangle, shew that their Jacobian, viz.

$$\begin{vmatrix} \frac{d\phi_1}{dx}, & \frac{d\phi_1}{dy}, & \frac{d\phi_1}{dz} \\ \frac{d\phi_2}{dx}, & \frac{d\phi_2}{dy}, & \frac{d\phi_2}{dz} \\ \frac{d\phi_3}{dx}, & \frac{d\phi_3}{dy}, & \frac{d\phi_3}{dz} \end{vmatrix},$$

equated to zero represents three straight lines, which are the sides of the self-conjugate triangle.

Hence, by considering two conics and their harmonic locus F (Art. 113, *post*), we can obtain the equations to the sides of the self-conjugate triangle.

97. Conics passing through the four angular points of a Quadrangle.

We have shewn in Art. 64 that, if we take as triangle of reference the triangle formed by three vertices of the quadrangle, then the coordinates of the angular points are proportional to (f, g, h), $(f, g, -h)$, $(f, -g, h)$, and $(-f, g, h)$.

The most general equation of a conic

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f_1\beta\gamma + 2g_1\gamma\alpha + 2h_1\alpha\beta = 0$$

is satisfied by these four points if

$$af^2 + bg^2 + ch^2 + 2f_1gh + 2g_1hf + 2h_1fg = 0,$$
$$af^2 + bg^2 + ch^2 - 2f_1gh - 2g_1hf + 2h_1fg = 0,$$
$$af^2 + bg^2 + ch^2 - 2f_1gh + 2g_1hf - 2h_1fg = 0,$$

and $$af^2 + bg^2 + ch^2 + 2f_1gh - 2g_1hf - 2h_1fg = 0.$$

Here clearly $f_1 = g_1 = h_1 = 0$, and the conic has its equation of the form $a\alpha^2 + b\beta^2 + c\gamma^2 = 0$, or, as in Art. 93, $L\alpha^2 + M\beta^2 + N\gamma^2 = 0$, where $Lf^2 + Mg^2 + Nh^2 = 0$.

Hence *The triangle formed by the three vertices of any complete quadrangle is self-conjugate with respect to all conics which pass through the vertices of the quadrangle.*

98. Conics touching the four sides of a Quadrilateral.

We have shewn in Art. 65 that, if we take as triangle of reference the triangle formed by the three diagonals of the quadrilateral, the equations of the sides can be written in the form

$$l\alpha \pm m\beta \pm n\gamma = 0.$$

The most general equation of a conic is

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \ldots\ldots(1),$$

and this touches these four lines, by Art. 72, if

$$Al^2 + Bm^2 + Cn^2 \pm 2Fmn \pm 2Gnl \pm 2Hlm = 0.$$

Hence, as in Art. 97, we have $F = 0$, $G = 0$, and $H = 0$.

$$\therefore\ gh - af = 0,\ hf - bg = 0,\ fg - ch = 0 \ldots\ldots(2).$$

Hence $$f = g = h = 0.$$

[Or $f = \pm\sqrt{bc}$; $g = \pm\sqrt{ca}$; $h = \pm\sqrt{ab}$, where all the ambiguities can be replaced by the positive sign, or two by the negative sign and one by the positive; in either case (1) is a perfect square and gives only two coincident straight lines.

Still another solution of (2) is $f=0$, $g=0$, $c=0$, in which case (1) reduces to a pair of straight lines through the vertex C.]

The only case admissible is therefore

$$a\alpha^2+b\beta^2+c\gamma^2=0, \text{ whence } bcl^2+cam^2+abn^2=0.$$

We shall use the form

$$L\alpha^2+M\beta^2+N\gamma^2=0 \text{ where } \frac{l^2}{L}+\frac{m^2}{M}+\frac{n^2}{N}=0.$$

Hence *The triangle formed by the three diagonals of a complete quadrilateral is self-conjugate with respect to all conics which touch the sides of the quadrilateral.*

99. Ex. 1. *Shew that the locus of the poles of a given straight line with regard to a system of conics passing through four given points is a conic.*

Let the four points be $(\pm f, \pm g, \pm h)$. The conic through them is

$$L\alpha^2+M\beta^2+N\gamma^2=0,$$

where $$Lf^2+Mg^2+Nh^2=0 \quad \text{.........................(1).}$$

The pole $(\alpha', \beta', \gamma')$ of a given straight line

$$l\alpha+m\beta+n\gamma=0 \quad \text{.........................(2)}$$

is given by $$\frac{L\alpha'}{l}=\frac{M\beta'}{m}=\frac{N\gamma'}{n}.$$

Substituting for L, M, N, in (1), we see that the locus of the required pole is

$$lf^2\beta\gamma+mg^2\gamma\alpha+nh^2\alpha\beta=0 \quad \text{.....................(3).}$$

This conic circumscribes the triangle of reference ABC (Fig., Art. 64).

Also, with the same figure, the equation to PQ is $\frac{\alpha}{f}+\frac{\beta}{g}=0$. The line joining the intersection of this with (2) to the point A is

$$(lf-mg)\beta=ng\gamma.$$

The fourth harmonic of this line with respect to the two lines AP, AQ, whose equations are $\frac{\beta}{\gamma}=\mp\frac{g}{h}$, is seen to be

$$nh^2\beta=g(lf-mg)\gamma,$$

and the intersection of this straight line with PQ, *i.e.* $\frac{\alpha}{f}+\frac{\beta}{g}=0$, is a point lying on (3).

Similarly it passes through the corresponding fourth harmonics to the points of intersection of the other five lines PR, PS, QR, QS, and RS with (2).

Finally, it must pass through the points on the given line where it is touched by the two conics of the system, for these points are poles of the straight line with respect to these two conics.

[These points will be found in Ex. 4 to be the double points of the involution determined on the given straight line by the system of conics.]

In all we have eleven points through which this conic passes. Hence it is known as the **eleven-point conic.**

Particular Case. Let $l\alpha+m\beta+n\gamma=0$ be the line at infinity so that $l=a$, $m=b$, $n=c$. The pole is now the centre, so that the locus of the centre of the conic is

$$af^2\beta\gamma+bg^2\gamma\alpha+ch^2\alpha\beta=0.$$

The fourth harmonics of the general case now become the middle points of the lines PQ, PR,.... For the middle point of PQ is the fourth harmonic of P, Q with respect to the intersection of PQ and the line at infinity.

Ex. 2. *Find the locus of the centre of the conics which touch four given straight lines.*

The straight lines being $l\alpha\pm m\beta\pm n\gamma=0$, as in Art. 98, the equation to the conic is

$$L\alpha^2+M\beta^2+N\gamma^2=0,$$

where

$$\frac{l^2}{L}+\frac{m^2}{M}+\frac{n^2}{N}=0 \text{ ....}(1).$$

The centre, being the pole of the line at infinity, is given by

$$\frac{L\bar{\alpha}}{a}=\frac{M\bar{\beta}}{b}=\frac{N\bar{\gamma}}{c}.$$

Hence, from (1), on substituting for L, M, N the locus of the centre is the straight line

$$\frac{l^2}{a}\alpha+\frac{m^2}{b}\beta+\frac{n^2}{c}\gamma=0.$$

By Art. 66 this is the straight line joining the middle points of the diagonals.

It follows at once, geometrically, that the middle point of a diagonal lies on the centre locus. For a very thin ellipse, practically coinciding with the straight line PR, satisfies the conditions of being a conic touching the four lines and its centre is the middle point of PR.

In a similar way we can shew that the locus of the pole of any given straight line with respect to the system of conics is a straight line, which passes through the point on each diagonal which with the intersection of the diagonal and the given straight line divides the diagonal harmonically.

Ex. 3. *If a conic be inscribed in a quadrilateral, shew that two of the sides of the quadrangle formed by its points of contact will pass through each of the angular points of the triangle formed by the diagonals of the quadrilateral, and will form a harmonic pencil with the two diagonals which meet these.*

Let the sides of the quadrilateral be $l\alpha\pm m\beta\pm n\gamma=0$, and the conic is then

$$L\alpha^2+M\beta^2+N\gamma^2=0,$$

where

$$\frac{l^2}{L}+\frac{m^2}{M}+\frac{n^2}{N}=0.$$

Let F, G, H, K be the points of contact of the sides PQ, QR, RS and SP. Then F is the point $\left(\frac{l}{L}, \frac{m}{M}, \frac{n}{N}\right)$, G is $\left(\frac{l}{L}, -\frac{m}{M}, \frac{n}{N}\right)$, H is $\left(-\frac{l}{L}, \frac{m}{M}, \frac{n}{N}\right)$, and K is $\left(\frac{l}{L}, \frac{m}{M}, -\frac{n}{N}\right)$. Hence the equations to FH and GK are

$$\frac{M\beta}{m} = \frac{N\gamma}{n} \text{ and } \frac{M\beta}{m} = -\frac{N\gamma}{n}.$$

These both pass through the point A and form a harmonic pencil with AB and AC, which are the two diagonals of the circumscribing quadrilateral which pass through A. (Fig., Art. 65.)

Similarly FK and GH pass through C, and FG and HK pass through B.

It follows that the diagonal-point triangle of the quadrangle $FGHK$ is the same as the diagonal-line triangle of the quadrilateral $PQRS$.

This again follows from the fact that a system of conics inscribed in a quadrilateral have one and only one common self-conjugate triangle.

Ex. 4. *A family of conics passes through four given points; shew that the pairs of points in which they are cut by any straight line form an involution system.* (Desargues' Theorem.)

Take the figure and notation of Art. 97.

Any conic of the family is given by

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0, \text{ where } Lf^2 + Mg^2 + Nh^2 = 0,$$

i.e. its equation is

$$L(\alpha^2 h^2 - f^2\gamma^2) + M(\beta^2 h^2 - g^2\gamma^2) = 0,$$

i.e. $$\alpha^2 h^2 - f^2\gamma^2 + \lambda(\beta^2 h^2 - g^2\gamma^2) = 0, \quad \text{..................(1)}$$

where λ is any constant.

Any straight line being

$$l\alpha + m\beta + n\gamma = 0 \quad \text{..........................(2)},$$

the equations to the lines joining the point A of the triangle of reference to its intersections with (1) is

$$h^2(m\beta + n\gamma)^2 - f^2 l^2\gamma^2 + \lambda l^2(\beta^2 h^2 - g^2\gamma^2) = 0.$$

By Art. 39, Cor., these lines for all values of λ give an involution pencil determined by the pair

$$\beta^2 h^2 - g^2\gamma^2 = 0,$$

and $$h^2(m\beta + n\gamma)^2 - f^2 l^2\gamma^2 = 0.$$

The first of these pairs gives the lines joining A to the intersections of (2) with PR, QS and the second gives the lines joining A to the intersections of (2) with PS and QR. Hence (2) cuts any conic of the system, including either of the pairs of straight lines through the four points, in pairs of points which are in involution.

It is easily seen that there are two conics of the system which touch the given line, and hence these two points of contact are the double points of the involution.

Ex. 5. *A system of conics touches four given straight lines; shew that the pairs of tangents which can be drawn from any point* $(\alpha', \beta', \gamma')$ *to them form a pencil of lines in involution.* (Correlative of Desargues' Theorem.)

With the notation of Art. 98, any conic touching the four given lines is

$$L\alpha^2+M\beta^2+N\gamma^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where

$$\frac{l^2}{L}+\frac{m^2}{M}+\frac{n^2}{N}=0,$$

i.e.

$$l^2MN+m^2NL+n^2LM=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The pairs of tangents from $(\alpha', \beta', \gamma')$ to (1) are

$$(L\alpha'^2+M\beta'^2+N\gamma'^2)(L\alpha^2+M\beta^2+N\gamma^2)=(L\alpha\alpha'+M\beta\beta'+N\gamma\gamma')^2 \quad (3).$$

The equation to the straight lines joining the point A of the triangle of reference to the intersections of (3) with BC is obtained by putting $\alpha=0$ in (3), and is thus

$$\frac{\beta^2\alpha'^2}{N}+\frac{\gamma^2\alpha'^2}{M}+\frac{(\beta\gamma'-\beta'\gamma)^2}{L}=0,$$

i.e., on eliminating L by means of (2), it is

$$l^2\gamma^2\alpha'^2-m^2(\beta\gamma'-\beta'\gamma)^2+\frac{M}{N}[l^2\beta^2\alpha'^2-n^2(\beta\gamma'-\beta'\gamma)^2]=0.$$

Hence by Art. 89, Cor., this gives a pencil of lines in involution which is determined by the pairs

$$l^2\gamma^2\alpha'^2-m^2(\beta\gamma'-\beta'\gamma)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

and

$$l^2\beta^2\alpha'^2-n^2(\beta\gamma'-\beta'\gamma)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

(4) is easily seen to be the lines joining A to the intersections with BC of the lines joining $(\alpha', \beta', \gamma')$ to P and R, and (5) the similar intersections for the points Q and S. This would be expected; for one conic touching the four given lines is a very thin ellipse almost exactly coinciding with the line PR, rounded off at P and R, and the tangents to this ellipse are clearly the straight lines joining $(\alpha', \beta', \gamma')$ to P and R. So two other conics of the system are the ellipses indefinitely nearly coincident with QS and O_1O_2. Hence the pairs of tangents from $(\alpha', \beta', \gamma')$ to all conics touching the four lines, including the line ellipses PR, QS, and O_1O_2 form an involution pencil.

It is clear that there are two conics of the system which pass through the point $(\alpha', \beta', \gamma')$, which are given by (2) and by

$$L\alpha'^2+M\beta'^2+N\gamma'^2=0.$$

Hence the two tangents to these two conics are the double lines of the involution pencil.

Cor. 1. The lines joining any point to the six vertices of the quadrilateral formed by four lines are in involution.

Cor. 2. *Deduce that, if a parabola is inscribed in a triangle, its directrix passes through the orthocentre of the triangle.*

Let the triangle be ABC, with O as its orthocentre, and let $\infty\infty'$ be the line at infinity. Then by the corollary one pair of lines in involution is OA, and the line joining O to the point of intersection of BC and $\infty\infty'$, *i.e.* OA and the line through O perpendicular to OA. So a second pair is OB and the line through O perpendicular to OB; and a third pair is OC and the line through O perpendicular to OC. Hence more than two pairs of the lines in involution are at right angles. Therefore (Art. 87) all the pairs of involution-lines are at right angles, *i.e.* from O all the pairs of tangents drawn to the parabolas are at right angles, and so O lies on all their directrices.

Ex. 6. *Shew that the director circles of all conics which touch four straight lines have a common radical axis which is the directrix of the parabola belonging to the system.*

The equation to any such conic is

$$L\alpha^2+M\beta^2+N\gamma^2=0,$$

where

$$\frac{l^2}{L}+\frac{m^2}{M}+\frac{n^2}{N}=0,$$

i.e.

$$l^2MN+m^2NL+n^2LM=0 \text{ ....(1).}$$

As in Art. 79, the equation to the director circle is

$$MN(\beta^2+\gamma^2+2\beta\gamma\cos A)+NL(\gamma^2+\alpha^2+2\gamma\alpha\cos B)$$
$$+LM(\alpha^2+\beta^2+2\alpha\beta\cos C)=0 \text{...(2).}$$

By (1) this passes through the points given by

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{l^2}=\frac{\gamma^2+\alpha^2+2\gamma\alpha\cos B}{m^2}=\frac{\alpha^2+\beta^2+2\alpha\beta\cos C}{n^2},$$

i.e., by Art. 88, Ex. 4, through the common points of the circles described on the three diagonals as diameters. Hence the circles (2) are coaxal for all values of L, M, N.

There is one parabola of the system, viz.

$$L_1\alpha^2+M_1\beta^2+N_1\gamma^2=0,$$

given by

$$\frac{l^2}{L_1}+\frac{m^2}{M_1}+\frac{n^2}{N_1}=0 \text{ and } \frac{a^2}{L_1}+\frac{b^2}{M_1}+\frac{c^2}{N_1}=0 \text{............(3),}$$

and its directrix, being a particular case of the above set of director circles, must pass through their common points.

In the case of the parabola the equation (2) must reduce to $(a\alpha+b\beta+c\gamma)\times$ equation of the directrix $=0$. Hence the equation of the directrix is

$$\frac{\alpha}{a}L_1(M_1+N_1)+\frac{\beta}{b}M_1(N_1+L_1)+\frac{\gamma}{c}N_1(L_1+M_1)=0,$$

i.e., from (3), by solving for L_1, M_1, N_1,

$$\frac{\alpha}{a}[l^2(b^2-c^2)-a^2(m^2-n^2)]+\ldots+\ldots=0,$$

and this is therefore the equation to the common radical axis of the system of director circles.

The above theorem may also be deduced from the involution theorem of Ex. 5.

Taking the figure of Art. 65, let O be either of the points of intersection of the circles on PR and QS as diameters, so that POR and QOS are right angles. Hence the involution pencil (Ex. 5, Cor. 1) has two pairs of corresponding rays at right angles. Hence, by Art. 37, all pairs of corresponding rays are at right angles. Hence O_1OO_2 is a right angle, and the circle on O_1O_2 as diameter passes through the points of intersection of the circles on PR and QS as diameters.

Also, if from O we draw a pair of tangents to any conic inscribed in the quadrangle formed by the four lines, these tangents form a pair of the line-involution system and are thus at right angles. Hence the director circle of this conic, and all other conics touching the four lines, pass through the point O, so that the director circles of all such conics are coaxal.

Since the director circles are coaxal their centres must lie on a line, *i.e.* the locus of the centre of all conics which touch four given straight lines is a straight line passing through the middle points of the diagonals of the quadrilateral formed by the four lines.

EXAMPLES.

1. If conics be drawn through four fixed points, prove that their polars with regard to a fixed point P pass through another fixed point Q; and that PQ is the tangent at P to that conic of the system which passes through P.

2. Prove that the envelope of the polar of a given point with respect to a system of conics inscribed in a given quadrilateral is a conic inscribed in the diagonal triangle of the quadrilateral. Shew that this last conic is a parabola if, and only if, the centre locus of the system passes through the given point.

3. A system of conics passes through four fixed points. Prove that the locus of the points of contact of tangents drawn to them from another fixed point is a cubic curve passing through the fixed point.

4. Shew that the polars with respect to two conics of a point on a given straight line intersect on a conic circumscribing the triangle which is self-conjugate with respect to the two conics.

5. From the proposition of Art. 99, Ex. 1, deduce that:

(1) the centre of a conic passing through the vertices of a triangle and the ortho-centre lies on the nine-point circle of the triangle, and

(2) the centre of a conic passing through the in-centre and the three e-centres of a triangle lies on the circum-circle of the triangle.

6. If on each diagonal of a complete quadrilateral there be taken a pair of points dividing the diagonal harmonically, the six points so obtained will lie on a conic.

7. Shew that the envelope of a straight line which is cut harmonically by two pairs of opposite sides of a quadrilateral is a conic touching the two sides of the quadrilateral.

8. Two conics, S and S', meet in A, B, C, D. If the tangents to S at A, B intersect on S', then the tangents at C, D to S also meet on S'.

9. If three conics circumscribe a quadrangle, a common tangent to any two is cut harmonically by the third. [The points of contact of the tangent are the double points of the involution determined by the tangent.]

10. If three conics are inscribed in a quadrilateral, the tangents to two of them at a common point, and the two tangents from that point to the third conic, form a harmonic pencil.

11. If two conics are inscribed in the same quadrilateral, shew that their tangents at a common point cut any diagonal harmonically.

12. If two quadrangles have the same harmonic triangle, their eight vertices lie on a conic, which under certain conditions becomes two straight lines.

13. If two quadrilaterals have the same harmonic triangle, their eight sides touch a conic, which under certain conditions reduces to two points.

14. If the ends of each of two diagonals of a complete quadrilateral are conjugate points with respect to a given conic, shew that the ends of the third diagonal are conjugate points also.

15. Shew that the harmonic triangle of the quadrangle formed by the common points of two conics coincides with the harmonic triangle of the quadrilateral formed by their common tangents. [Two conics have only one common self-conjugate triangle.]

16. If chords PQ, QR, RS of a conic pass through three fixed collinear points, the chord PS passes through a fixed point. [Use Desargues' Theorem.]

17. The envelope of the polar of a fixed point P with regard to a system of conics touching four given straight lines is a conic which touches (1) the three sides of the self-conjugate triangle of the conics, (2) the six lines obtained by joining P to any point of intersection, Q, of the four lines and taking the harmonic conjugate of this line for the two common tangents through Q, and (3) the tangents at P to the two conics of the systems which pass through P. [**Eleven-Line Conic.**]

18. From a fixed point O in the third diagonal of a quadrilateral $PQRS$ tangents are drawn to a series of conics inscribed in the quadrilateral. Shew that the locus of their points of contact is a conic which passes through P, Q, R, and S.

19. Shew that the cross-ratio of the pencil (or range) formed by the polars (or poles) of a point (or line) P with respect to a system of four-point (or four-line) conics is the same for all positions of P.

20. Shew that the locus of the foci of conics inscribed in the quadrilateral $l\alpha \pm m\beta \pm n\gamma = 0$ is the cubic

$$(a\alpha+b\beta+c\gamma)(l^2\alpha^2\cot A+m^2\beta^2\cot B+n^2\gamma^2\cot C)$$
$$=(a\beta\gamma+b\gamma\alpha+c\alpha\beta)\left(\frac{l^2\alpha}{\sin A}+\frac{m^2\beta}{\sin B}+\frac{n^2\gamma}{\sin C}\right).$$

[This cubic passes through the circular points at infinity and through the six intersections of the four given lines.]

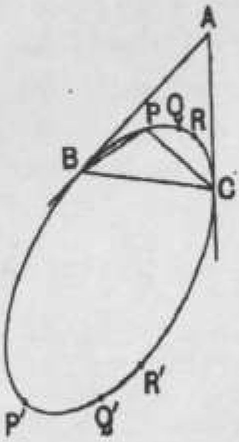
21. A conic touches the sides PQ, QR, RS, SP of a quadrilateral $PQRS$ and meets any straight line through a diagonal point in T and T'; prove that P, Q, R, S, T and T' lie on a conic.

22. If p, q are the perpendiculars from opposite vertices of a quadrilateral, and r, s those from the other pair of vertices, upon any tangent to a conic inscribed in the quadrilateral, prove that $pr = kqs$, where k is a constant.

If the conic be a parabola, shew that $k=1$.

100. Conic referred to a triangle given by two tangents and their chord of contact.

Let the vertices, B and C, of the triangle of reference be two points on a conic, and let the tangents at B and C meet in the third vertex A.



Then, as in Part I, Art. 409, the equation to the conic may be written in the form

$$\beta\gamma = k\alpha^2.$$

The coordinates of any point on this conic may be taken to be (t, k, t^2). For these quantities clearly satisfy the equation.

The fact that any point on the curve can be so simply given in terms of one variable parameter makes this form of the equation very useful in many cases.

The point (t, k, t^2) may be called the point "t."

The equation to the straight line joining the points "t" and "t'" is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ t, & k, & t^2 \\ t', & k, & t'^2 \end{vmatrix} = 0,$$

i.e. $$k\alpha(t'+t) - \beta tt' - k\gamma = 0, \ldots\ldots\ldots\ldots\ldots(1),$$

on dividing by the factor $t' - t$.

Putting $t' = t$ in (1) we have as the equation to the tangent at the point "t"

$$2k\alpha t - \beta t^2 - k\gamma = 0.$$

The tangents at the points "t" and "t_1" meet at the point $(t + t_1,\ 2k,\ 2tt_1)$.

The straight line $l\alpha + m\beta + n\gamma = 0$ is a tangent if

$$\frac{2kt}{l} = \frac{-t^2}{m} = \frac{-k}{n},$$

i.e. if $l^2 = 4kmn$.

Hence the conic is a parabola if $a^2 = 4kbc$.

101. If P be the point "t," the equation to BP is $t\alpha - \gamma = 0$, and that to CP is $k\alpha - \beta t = 0$.

It follows that, if t is given, the positions of the lines BP and CP are both unique, and hence P is known uniquely; similarly, when P is given, t is also definitely given. Between "t" and P there is thus a one-one correspondence.

If t be positive, the point P is on the part of the curve which is on the same side of BC as A is; if it be negative, then P is on the other side of BC, as at P' in the figure. When t is zero, P is at the point B; when t is infinite P is at C; for the equation of BP, which is $\alpha - \frac{\gamma}{t} = 0$, then becomes $\alpha = 0$, *i.e.* BP and BC then coincide, so that P is then at C.

102. Ex. 1. *The base of a triangle touches a given conic; its extremities move on two fixed tangents to the conic, and the other sides of the triangle pass through fixed points; shew that the locus of the vertex is a conic passing through the given points.*

Let the points of contact of the two fixed tangents be B and C, and let them intersect in A. Take ABC as the triangle of reference, so that the equation to the conic is $\beta\gamma = k\alpha^2$.

Let the variable triangle be LMN, where M lies on AC and N on AB.

Since MN touches the conic, its equation is

$$2kt\alpha - t^2\beta - k\gamma = 0.$$

Hence M is the point $(1, 0, 2t)$, and N is the point $(t, 2k, 0)$.

Let the fixed points through which LM and LN pass be $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$.

The equation to LM is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \\ 1, & 0, & 2t \end{vmatrix} = 0,$$

i.e. $$2t(\alpha\beta_1 - \alpha_1\beta) = -(\beta\gamma_1 - \beta_1\gamma).$$

So the equation to LN is

$$t(\beta\gamma_2 - \beta_2\gamma) = 2k(\alpha\gamma_2 - \alpha_2\gamma).$$

Hence, on eliminating t, we have, as the locus of L,

$$(\beta\gamma_1 - \beta_1\gamma)(\beta\gamma_2 - \beta_2\gamma) + 4k(\alpha\beta_1 - \alpha_1\beta)(\alpha\gamma_2 - \alpha_2\gamma) = 0,$$

which is a conic passing through the two fixed points.

Ex. 2. *Shew that the equation to the circle of curvature at the point B of the conic $\beta\gamma = k\alpha^2$ is*

$$kb(\alpha^2 + \gamma^2 + 2\alpha\gamma\cos B) = \gamma(a\alpha + b\beta + c\gamma),$$

and that the radius of curvature is $\dfrac{R\sin A\sin C}{k\sin^2 B}$, where R is the radius of the circum-circle of the triangle of reference.

The tangent at B is $\gamma = 0$. Hence by Part I, Art 385, III, the equation to a conic having contact of the second order with the given conic at B is

$$\beta\gamma - k\alpha^2 + \gamma(\lambda\alpha + \mu\gamma) = 0,$$

i.e. $$-k\alpha^2 + \mu\gamma^2 + \beta\gamma + \lambda\gamma\alpha = 0.$$

This is a circle if

$$\mu b^2 - bc = \mu a^2 - kc^2 - \lambda ca = -kb^2. \qquad \text{(Art. 85.)}$$

$$\therefore \mu b = c - bk \text{ and } \lambda b = a - 2kb\cos B.$$

Hence the required equation is

$$kb(\alpha^2 + \gamma^2 + 2\alpha\gamma\cos B) = \gamma(a\alpha + b\beta + c\gamma) \text{ ....(1).}$$

But, if P be any point on the circle of curvature, we have BP^2 = diameter of the circle × perpendicular from P on the tangent at B, so that, by Art. 49, the equation to the circle is

$$\frac{\alpha^2 + \gamma^2 + 2\alpha\gamma\cos B}{\sin^2 B} = 2\rho\gamma = \frac{\rho}{\Delta}(a\alpha + b\beta + c\gamma)\gamma.$$

Comparing this with (1), we have

$$\rho = \frac{\Delta}{kb\sin^2 B} = \frac{R\sin A\sin C}{k\sin^2 B}.$$

EXAMPLES.

1. The base of a triangle passes through a fixed point, and the extremities of the base move on fixed tangents to a conic, whilst the remaining sides touch the conic. Shew that the locus of the vertex is a conic having double contact with the given conic.

2. The sides of a triangle touch the conic $\beta\gamma=k\alpha^2$, and two of its vertices lie on the straight lines $\beta-\lambda_1\gamma=0$ and $\beta-\lambda_2\gamma=0$ respectively; prove that the locus of the third vertex is

$$(\lambda_1+\lambda_2)^2\beta\gamma=4k\lambda_1\lambda_2\alpha^2.$$

3. The intersections of a straight line with a conic and its intersections with a pair of fixed tangents to the conic form a range of constant cross-ratio; shew that its envelope is a conic having double contact with the given conic.

4. Given two points, B and C, on a conic, shew that the envelope of a chord PQ, such that B, P, C, Q give a constant cross-ratio when joined to any point of the given conic, is a conic touching the given conic at B and C.

5. A triangle is inscribed in a conic and two of its sides pass through fixed points M, N; shew that the third side envelopes a conic having double contact with the given conic at its intersections with the line MN.

6. A chord APP' of a conic is drawn through a fixed point A, to meet the curve in P and P', and on it is taken a point Q such that $(APQP')$ is constant; shew that the locus of Q is a conic having double contact with the given one.

7. PP', QQ' and RR' are chords of a conic which meet in a point A, and S is any other point on the conic. Shew that the points of intersection of QR and SP', RP and SQ', PQ and SR' lie on a straight line passing through A.

8. A family of conics have double contact at two given points B and C; shew that the locus of the points of contact of tangents drawn in a given direction is a conic passing through B, C and the intersection of the tangents at B and C.

9. Two triangles are formed each by two tangents to a conic and their chord of contact; prove that their angular points lie on a conic, and that their six sides touch another conic.

10. Find, in areal coordinates, the equation of the parabola, which touches the sides AB, AC of the fundamental triangle at B and C, and shew that its focus is the point $(b^2+c^2-a^2, b^2, c^2)$.

108. Ex. 1. *If two triangles ABC, $A'B'C'$ are inscribed in a conic, their sides touch a conic, and they are self-conjugate with respect to a third conic.*

Take ABC as the triangle of reference, and let A', B', C' be the points

$$(\alpha_1, \beta_1, \gamma_1),\ (\alpha_2, \beta_2, \gamma_2) \text{ and } (\alpha_3, \beta_3, \gamma_3).$$

Let the first conic be

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0 \text{ ....(1).}$$

Then, as in Art. 90, the equation to $B'C'$ is

$$\frac{L\alpha}{\alpha_2\alpha_3} + \frac{M\beta}{\beta_2\beta_3} + \frac{N\gamma}{\gamma_2\gamma_3} = 0 \text{ ....(2).}$$

Any conic inscribed in ABC is

$$\sqrt{L_1\alpha} + \sqrt{M_1\beta} + \sqrt{N_1\gamma} = 0 \text{ ....(3);}$$

and, by Art. 91, (2) touches (3) if

$$L_1\frac{MN}{\beta_2\beta_3\gamma_2\gamma_3} + M_1\frac{NL}{\gamma_2\gamma_3\alpha_2\alpha_3} + N_1\frac{LM}{\alpha_2\alpha_3\beta_2\beta_3} = 0,$$

and this is satisfied by

$$L_1 = \frac{L^2}{\alpha_1\alpha_2\alpha_3},\ M_1 = \frac{M^2}{\beta_1\beta_2\beta_3} \text{ and } N_1 = \frac{N^2}{\gamma_1\gamma_2\gamma_3},$$

since the point A' lies on (1).

Hence $B'C'$ touches the conic

$$L\sqrt{\frac{\alpha}{\alpha_1\alpha_2\alpha_3}} + M\sqrt{\frac{\beta}{\beta_1\beta_2\beta_3}} + N\sqrt{\frac{\gamma}{\gamma_1\gamma_2\gamma_3}} = 0,$$

and, by symmetry, the conic will be touched by $C'A'$, $A'B'$ also.

Again, any conic self-conjugate with regard to ABC is

$$L_2\alpha^2 + M_2\beta^2 + N_2\gamma^2 = 0,$$

and $(\alpha_1, \beta_1, \gamma_1)$, *i.e.* A', is the pole of $B'C'$ with respect to this conic if

$$\frac{L_2\alpha_1}{\frac{L}{\alpha_2\alpha_3}} = \frac{M_2\beta_1}{\frac{M}{\beta_2\beta_3}} = \frac{N_2\gamma_1}{\frac{N}{\gamma_2\gamma_3}},$$

i.e. for the conic $\dfrac{L\alpha^2}{\alpha_1\alpha_2\alpha_3} + \dfrac{M\beta^2}{\beta_1\beta_2\beta_3} + \dfrac{N\gamma^2}{\gamma_1\gamma_2\gamma_3} = 0,$

and so, by symmetry, B' and C' are the poles of $C'A'$ and $A'B'$ with respect to it.

It follows that all triangles inscribed in the conic

$$L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0,$$

and touching the conic $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0$, are self-conjugate with respect to the conic

$$\frac{l}{L}\alpha^2 + \frac{m}{M}\beta^2 + \frac{n}{N}\gamma^2 = 0.$$

Ex. 2. *If the sides of two triangles ABC, A′B′C′ touch a conic, their angular points lie on a conic, and they are self-conjugate with respect to a third conic.*

Take ABC as the triangle of reference, so that the equation of the first conic is

$$\sqrt{L\alpha}+\sqrt{M\beta}+\sqrt{N\gamma}=0 \quad \text{......................(1)}.$$

Let the coordinates of the points of contact of $B'C'$, $C'A'$ and $A'B'$ be

$$(\alpha_1, \beta_1, \gamma_1),\ (\alpha_2, \beta_2, \gamma_2) \text{ and } (\alpha_3, \beta_3, \gamma_3).$$

Then, as in Art. 91, the equations to $B'C'$ and $C'A'$ are

$$\alpha\sqrt{\frac{L}{\alpha_1}}+\beta\sqrt{\frac{M}{\beta_1}}+\gamma\sqrt{\frac{N}{\gamma_1}}=0$$

and
$$\alpha\sqrt{\frac{L}{\alpha_2}}+\beta\sqrt{\frac{M}{\beta_2}}+\gamma\sqrt{\frac{N}{\gamma_2}}=0.$$

These meet at the point $(\sqrt{\alpha_1\alpha_2},\ \sqrt{\beta_1\beta_2},\ \sqrt{\gamma_1\gamma_2})$, since the coordinates of the points of contact of $B'C'$, $C'A'$ satisfy (1).

So the points of intersection A' and B' of the other tangents are

$$(\sqrt{\alpha_2\alpha_3},\ \sqrt{\beta_2\beta_3},\ \sqrt{\gamma_2\gamma_3}) \text{ and } (\sqrt{\alpha_3\alpha_1},\ \sqrt{\beta_3\beta_1},\ \sqrt{\gamma_3\gamma_1}).$$

These all lie on the conic

$$\sqrt{L\alpha_1\alpha_2\alpha_3}\,\beta\gamma+\sqrt{M\beta_1\beta_2\beta_3}\,\gamma\alpha+\sqrt{N\gamma_1\gamma_2\gamma_3}\,\alpha\beta=0,$$

which is a conic circumscribing the triangle ABC.

As in the preceding example, it is easily seen that both triangles are self-conjugate with respect to the conic

$$\sqrt{\frac{L}{\alpha_1\alpha_2\alpha_3}}\,\alpha^2+\sqrt{\frac{M}{\beta_1\beta_2\beta_3}}\,\beta^2+\sqrt{\frac{N}{\gamma_1\gamma_2\gamma_3}}\,\gamma^2=0.$$

It follows that all conics circumscribing the conic

$$\sqrt{L\alpha}+\sqrt{M\beta}+\sqrt{N\gamma}=0,$$

and inscribed in the conic

$$l\beta\gamma+m\gamma\alpha+n\alpha\beta=0,$$

are self-conjugate with respect to the conic

$$\frac{L}{l}\alpha^2+\frac{M}{m}\beta^2+\frac{N}{n}\gamma^2=0.$$

Ex. 3. *If two triangles ABC, A′B′C′ are self-conjugate with regard to a conic, their six vertices lie on a second conic and their six sides touch a third conic.*

Let ABC be the triangle of reference, so that the first conic is

$$l\alpha^2+m\beta^2+n\gamma^2=0 \quad \text{..........................(1)}.$$

Let A', B', C' be the points

$$(\alpha_1, \beta_1, \gamma_1),\ (\alpha_2, \beta_2, \gamma_2),\ (\alpha_3, \beta_3, \gamma_3).$$

Since each is the pole of the line joining the other two, we have

$$l\alpha_2\alpha_3 + m\beta_2\beta_3 + n\gamma_2\gamma_3 = 0 \text{ ....(2)},$$

$$l\alpha_3\alpha_1 + m\beta_3\beta_1 + n\gamma_3\gamma_1 = 0 \text{ ....(3)},$$

and

$$l\alpha_1\alpha_2 + m\beta_1\beta_2 + n\gamma_1\gamma_2 = 0 \text{ ....(4)}.$$

But these are the conditions that the points A', B', C' lie on the conic

$$\frac{l\alpha_1\alpha_2\alpha_3}{\alpha} + \frac{m\beta_1\beta_2\beta_3}{\beta} + \frac{n\gamma_1\gamma_2\gamma_3}{\gamma} = 0 \text{ ....(5)},$$

which clearly passes through A, B, and C.

Again the equation to $B'C'$, the polar of A' with respect to (1), is

$$l\alpha\alpha_1 + m\beta\beta_1 + n\gamma\gamma_1 = 0,$$

and this touches the conic

$$\sqrt{L\alpha} + \sqrt{M\beta} + \sqrt{N\gamma} = 0,$$

if, by Art. 91, $\quad Lmn\beta_1\gamma_1 + Mnl\gamma_1\alpha_1 + Nlm\alpha_1\beta_1 = 0,$

which is satisfied if

$$L = l^2\alpha_1\alpha_2\alpha_3,\ M = m^2\beta_1\beta_2\beta_3 \text{ and } N = n^2\gamma_1\gamma_2\gamma_3,$$

since (2) is true.

Hence $B'C'$ touches the conic

$$l\sqrt{\alpha_1\alpha_2\alpha_3\alpha} + m\sqrt{\beta_1\beta_2\beta_3\beta} + n\sqrt{\gamma_1\gamma_2\gamma_3\gamma} = 0 \text{ ....(6)},$$

and so, by symmetry, do $C'A'$ and $A'B'$. Also this conic is touched by the sides of the triangle ABC.

Ex. 4. *If two conics S_1, S_2 are such that one triangle can be inscribed in S_1 and circumscribed about S_2, then an infinite number of such triangles can be drawn.* (Poncelet's Theorem.)

Let ABC be the given triangle which is inscribed in S_1 and circumscribed about S_2. Draw any other tangent to S_2 to cut S_1 in B' and C', and from B', C' draw the other tangents to S_2 to meet in A'. By Ex. 2, since the sides of the triangles ABC, $A'B'C'$ touch a conic, their angular points lie on a conic, *i.e.* A' lies on the conic determined by the five points A, B, C, B', C', *i.e.* A' lies on S_1. So for any other tangent $B''C''$ drawn to S_2.

Ex. 5. *If two conics S_1, S_2 are such that one triangle which is self-conjugate for S_1 can be inscribed in (or circumscribed to) S_2, then an infinite number of triangles can be so described.*

[Let ABC be the given triangle; let A' be any point on S_2 and let its polar with regard to S_1 meet S_2 in B' and C'; also let the polar, $A'C''$, of B' meet $B'C'$ in C''; apply Ex. 3 to the two triangles ABC, $A'B'C''$, which are self-conjugate with respect to S_1. Similarly, for the second part, by starting with any tangent $B'C'$ to S_2.]

104. Areal Coordinates. Much of the preceding Chapter holds for Areal Coordinates as well as trilinears; the chief differences are set down here.

The conic represented by the general equation

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is a parabola if (Art. 74)

$$A + B + C + 2F + 2G + 2H = 0;$$

it is a circle if

$$\frac{b + c - 2f}{a_0^2} = \frac{c + a - 2g}{b_0^2} = \frac{a + b - 2h}{c_0^2} \quad \text{(Art. 85)};$$

and it is a rectangular hyperbola if

$$aa_0^2 + bb_0^2 + cc_0^2 - 2fb_0c_0 \cos A - 2gc_0a_0 \cos B - 2ha_0b_0 \cos C = 0. \quad \text{(Art. 80.)}$$

Its centre is given by

$$ax_1 + hy_1 + gz_1 = hx_1 + by_1 + fz_1 = gx_1 + fy_1 + cz_1,$$

i.e.

$$\frac{d\phi}{ax_1} = \frac{d\phi}{ay_1} = \frac{d\phi}{az_1},$$

and is thus the point

$$(A + H + G,\ H + B + F,\ G + F + C). \quad \text{(Art. 78.)}$$

The equation to the circum-circle is

$$a_0^2 yz + b_0^2 zx + c_0^2 xy = 0. \quad \text{(Art. 83.)}$$

The equation to any circle is

$$a_0^2 yz + b_0^2 zx + c_0^2 xy = (x + y + z)(t_1^2 x + t_2^2 y + t_3^2 z),$$

where t_1, t_2, t_3 are the lengths of the tangents to the circle from the points A, B, C of the triangle of reference (Art. 86).

That of the in-circle is

$$\sqrt{x \cot \frac{A}{2}} + \sqrt{y \cot \frac{B}{2}} + \sqrt{z \cot \frac{C}{2}} = 0. \quad \text{(Art. 87.)}$$

The e-circle opposite A is

$$\sqrt{-x \cot \frac{A}{2}} + \sqrt{y \tan \frac{B}{2}} + \sqrt{z \tan \frac{C}{2}} = 0. \quad \text{(Art. 87.)}$$

The nine-point circle is

$$b_0c_0 \cos Ax^2 + c_0a_0 \cos By^2 + a_0b_0 \cos Cz^2 - a_0^2 yz - b_0^2 zx - c_0^2 xy = 0. \quad \text{(Art. 87.)}$$

The self-conjugate circle is

$$x^2 \cot A + y^2 \cot B + z^2 \cot C = 0. \qquad \text{(Art. 87.)}$$

The foci of the general conic are given by

$$\frac{4(b+c-2f)\,\phi(x, y, z) - \left(\dfrac{d\phi}{dy} - \dfrac{d\phi}{dz}\right)^2}{a_0^2}$$

$$= \text{two similar expressions.} \qquad \text{(Art. 89.)}$$

In general if the trilinear equation

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

represents the same conic as the areal equation

$$a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy = 0,$$

then, since

$$\frac{x}{a_0\alpha} = \frac{y}{b_0\beta} = \frac{z}{c_0\gamma},$$

we have

$$\frac{a_1a_0^2}{a} = \frac{b_1b_0^2}{b} = \frac{c_1c_0^2}{c} = \frac{f_1b_0c_0}{f} = \frac{g_1c_0a_0}{g} = \frac{h_1a_0b_0}{h},$$

where a_0, b_0, c_0 are the sides of the triangle of reference.

These relations enable us to transform the coefficients in a trilinear equation to the corresponding coefficients in an areal equation, and conversely.

EXAMPLES.

1. Shew that the polars of the vertices A, B, C of a triangle with respect to any conic meet the opposite sides in three collinear points, and that the lines joining A, B, C to the poles of the opposite sides are concurrent.

2. Three points are taken on the sides BC, CA, AB of a triangle such that $\frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB}$; prove that each of the sides of the triangle PQR touches a fixed parabola.

3. Determine all the common tangents of the conics

$$y^2 + z^2 + 2yz + 2zy = 0$$

and

$$x^2 + y^2 + 9z^2 - 6yz - 6zx - 8xy = 0.$$

4. Shew that the equation

$$\sqrt{(m-n)(y-z)} + \sqrt{(n-l)(z-x)} + \sqrt{(l-m)(x-y)} = 0$$

represents only a pair of coincident straight lines.

5. Shew that the pair of tangents that can be drawn to the conic

$$2(\beta\cos B-\alpha\cos A)\gamma-\beta^2\cos C=0$$

with the straight line $\alpha=0$ as the chord of contact are at right angles if $A=B$.

6. Shew how to find the condition that two given conics may intersect in four concyclic points. Prove that, if a, b, c are the sides of the triangle of reference, the conics

$$\beta\gamma+\gamma\alpha+\alpha\beta=0$$

and $a(b+c)\alpha^2+b(c+a)\beta^2+c(a+b)\gamma^2-2bc\beta\gamma-2ca\gamma\alpha-2ab\alpha\beta=0$

satisfy this condition, and find the equation to the circle through their points of intersection.

7. Shew that the general equation in areal coordinates of a conic which passes through the middle points of the sides of the triangle of reference is

$$fx(x+y-z)+gy(z+x-y)+hz(y+z-x)=0.$$

Find the ratios of f, g, h that the equation may be that of the nine-point circle of the triangle.

8. Shew that the three pairs of straight lines

$$a_1\alpha^2+2b_1\alpha\beta+c_1\beta^2=0,\quad a_2\beta^2+2b_2\beta\gamma+c_2\gamma^2=0,$$

and

$$a_3\gamma^2+2b_3\gamma\alpha+c_3\alpha^2=0$$

will touch the same conic if $a_1a_2a_3=c_1c_2c_3$.

Shew that the three perpendiculars from the angular points of a triangle upon the opposite sides and the three medians all touch a conic.

9. Prove that the lines joining each angular point of a triangle to the points in which the opposite side is cut by a conic all touch another conic.

10. Tangents drawn from the angular points of the triangle of reference to the conic

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0$$

meet the opposite sides in six points; shew that these six points lie on the conic

$$BC\alpha^2+CA\beta^2+AB\gamma^2-2AF\beta\gamma-2BG\gamma\alpha-2CH\alpha\beta=0,$$

where A, B,... have the usual meaning.

Shew also that the above two conics and the conic

$$\sqrt{\alpha(gh-af)}+\sqrt{\beta(hf-bg)}+\sqrt{\gamma(fg-ch)}=0$$

have a common inscribed quadrilateral.

11. If two straight lines are conjugate with respect to a conic, shew that they, and the two tangents drawn to the conic from their point of intersection, form a harmonic pencil.

12. Through two fixed points are drawn two lines conjugate with respect to a given conic. Shew that the locus of their point of intersection is a conic, passing through the fixed points and through the points of intersection of their polars with the conic.

13. Each angular point of the triangle of reference is joined to two fixed points $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$; shew that their six points of intersection with the opposite sides of the triangle lie on the conic

$$\frac{\alpha^2}{\alpha_1\alpha_2}+\frac{\beta^2}{\beta_1\beta_2}+\frac{\gamma^2}{\gamma_1\gamma_2}-\beta\gamma\left[\frac{1}{\beta_1\gamma_2}+\frac{1}{\beta_2\gamma_1}\right]$$
$$-\gamma\alpha\left[\frac{1}{\gamma_1\alpha_2}+\frac{1}{\gamma_2\alpha_1}\right]-\alpha\beta\left[\frac{1}{\alpha_1\beta_2}+\frac{1}{\alpha_2\beta_1}\right]=0.$$

14. If two triangles ABC, $A'B'C'$ are such that A, B, C are the poles of $B'C'$, $C'A'$, $A'B'$ with respect to a conic, shew that the triangles are in perspective. Prove also that their centre of perspective is the pole of the axis of perspective with regard to the conic.

15. A straight line drawn through a fixed point A cuts a conic passing through two fixed points B and C in P and the line BC in A_1; on it is taken a point Q such that the cross-ratio of A, P, A_1, Q is constant; shew that the locus of Q is a conic through B and C.

16. Prove Carnot's theorem, that if a conic meets the sides BC, CA, AB of a triangle in A_1 and A_2, B_1 and B_2, C_1 and C_2, then

$$AB_1 . AB_2 . BC_1 . BC_2 . CA_1 . CA_2 = CB_1 . CB_2 . AC_1 . AC_2 . BA_1 . BA_2.$$

[Use Areal Coordinates.]

17. Obtain the condition that the general equation in trilinear coordinates shall represent a rectangular hyperbola from the fact that its intersections with the line at infinity form with the circular points at infinity a harmonic range, so that these circular points are conjugate with respect to the conic.

18. In Areal Coordinates shew that the tangential equations to the circum-circle, in-circle, self-polar circle and nine-point circle of the triangle of reference are respectively

$$a\sqrt{p}+b\sqrt{q}+c\sqrt{r}=0,$$
$$(b+c-a)\,qr+(c+a-b)\,rp+(a+b-c)\,pq=0,$$
$$p^2\tan A+q^2\tan B+r^2\tan C=0,$$

and
$$a\sqrt{q+r}+b\sqrt{r+p}+c\sqrt{p+q}=0,$$

where a, b, c are the sides of the triangle of reference.

19. Shew that the equation of the circle of curvature at the point A of the conic $L\beta\gamma+M\gamma\alpha+N\alpha\beta=0$ is

$$aLMN(\beta^2+\gamma^2+2\beta\gamma\cos A)$$
$$=(M^2+N^2-2MN\cos A)(N\beta+M\gamma)(a\alpha+b\beta+c\gamma),$$

and that the corresponding radius of curvature is

$$R\frac{bc}{a^2}\frac{(M^2+N^2-2MN\cos A)^{\frac{3}{2}}}{LMN},$$

where R is the circum-radius of the triangle of reference.

20. Shew that the radical centre of the escribed circles of a triangle is the centre of the inscribed conic whose equation in areal coordinates is $\sqrt{ax}+\sqrt{by}+\sqrt{cz}=0$.

21. If two conics have each double contact with a third, their chords of contact with the third conic, and a pair of their chords of intersection with each other, all pass through the same point and form a harmonic pencil.

22. If three conics have double contact with a fourth, six of their chords of intersection pass, three by three, through the same point, and form the sides and diagonals of a complete quadrilateral.

23. Two triangles are inscribed in a given triangle and are in perspective with it; shew that a conic self-conjugate with respect to the given triangle can be drawn to touch their six sides.

If the two triangles are circumscribed to the given triangle and are in perspective with it then, similarly, a conic, self-conjugate with regard to the given triangle, can be drawn through their six vertices.

24. If the lines joining the vertices of the triangle of reference to those of the points in which the conic

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0$$

meets the opposite sides are concurrent, the same is true for the other two points, and the condition for this is

$$abc-2fgh-af^2-bg^2-ch^2=0.$$

[Let three of the lines be $\beta-p\gamma=0$, $\gamma-q\alpha=0$, and $\alpha-r\beta=0$, so that $pqr=1$. Since $\beta-p\gamma=0$ is a factor of

$$b\beta^2+2f\beta\gamma+c\gamma^2=0,\ \therefore -2f=bp+\frac{c}{p},$$

and so for the others,

$$\therefore -8fgh=\left(bp+\frac{c}{p}\right)\left(cq+\frac{a}{q}\right)\left(ar+\frac{b}{r}\right)$$

$$=2abc+\frac{ac^2}{p^2}+\frac{ba^2}{q^2}+\frac{cb^2}{r^2}+ab^2p^2+bc^2q^2+ca^2r^2,\text{ since } pqr=1,$$

$$=-4abc+a\left(\frac{c}{p}+bp\right)^2+\ldots+\ldots=-4abc+4af^2+4bg^2+4ch^2.]$$

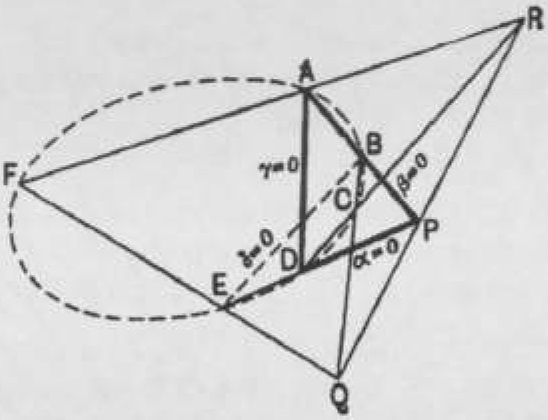
CHAPTER IV

MISCELLANEOUS THEOREMS

[This Chapter may be omitted by the Student on a first reading.]

105. Pascal's Theorem. *If a hexagon be inscribed in a conic, the points of intersection of the three pairs of opposite sides are collinear.*

Let $ABCDEF$ be the hexagon; let AB and DE meet in P, BC and EF in Q, and CD and FA in R. Then PQR shall be a straight line.



Take ADP as the triangle of reference, so that the equations to DP, PA, AD are respectively $\alpha=0$, $\beta=0$ and $\gamma=0$.

Let $\delta \equiv l\alpha+m\beta+n\gamma=0$ be the equation of BE.

Then the equation to the conic is

$$\alpha\beta=k\gamma\delta \qquad\text{.........................(1).}$$

Let the equation to CD be $\alpha=\lambda\gamma$, so that, by (1), the equation to BC is

$$\lambda\beta=k\delta.$$

Let the equation to AF be $\beta=\mu\gamma$, so that similarly the equation to EF is

$$\mu\alpha=k\delta.$$

Then P is given by

$$\beta=0 \text{ and } \alpha=0;$$

Q is given by

$$\lambda\beta = k\delta \text{ and } \alpha\mu = k\delta;$$

and R is given by

$$\alpha = \lambda\gamma \text{ and } \beta = \mu\gamma.$$

All three points lie on the straight line whose equation is

$$\alpha\mu = \lambda\beta.$$

106. Brianchon's Theorem. *If a hexagon circumscribe a conic, the lines joining its opposite vertices meet in a point.*

Take the figure of the previous article. Let the tangents at A, B meet in X; the tangents at B, C in Y; those at C, D in Z; those at D, E in U; those at E, F in V; and those at F, A in W.

Since the polars of X and U, *i.e.* AB and DE, pass through the point P, therefore the polar of P passes through X and U, *i.e.* the polar of P is XU.

So the polars of Q, R are YV and ZW.

But the points P, Q, R have been shewn to lie on a line. Hence, (Art. 12), their polars meet in a point, *i.e.* the three diagonals of the hexagon $XYZUVW$ meet in a point.

107. By taking the points A, B, C, D, E, F on the conic in different orders we shall obtain sixty different hexagons. For if we keep one of them, A, fixed we can permute the other five points in $\underline{|5}$ different ways, and the number of different hexagons is obtained by dividing this number by 2. For any such permutation as $BFDEC$, when joined to A, gives the same hexagon as it does when reversed as $CEDFB$, since the hexagon $ABFDEC$ is the same figure as $ACEDFB$. Hence altogether there are sixty such hexagons, and sixty Pascal lines.

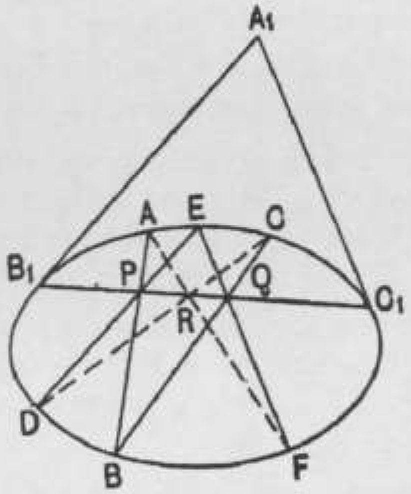
These Pascal lines pass three by three through twenty points. To prove this, let the alternate sides of one of the hexagons be produced to form two triangles. Thus, taking the hexagon $ABCDEF$, let FA, CB meet in X, FA and DE in Y, DE and BC in Z, CD and EF in X', AB and CD in Y' and finally AB and EF in Z'.

Then, since by Pascal's Theorem the corresponding sides of the triangles XYZ, $X'Y'Z'$ meet in collinear points, they are coaxal and therefore copolar, *i.e.* XX', YY' and ZZ' are concurrent.

But XX', YY', ZZ' are respectively the Pascal lines of the hexagons $EFADCB$, $BAFCDE$, and $FEDABC$.

Similarly for six tangents to a conic there are sixty Brianchon points, which lie three by three on straight lines.

108. When the angular points A, B, C, ... of a Pascal hexagon are so arranged that the Pascal line cuts the conic, an easy analytical proof may be obtained for both Pascal's and Brianchon's Theorems.



Let the line joining the intersections of AB and DE, and of BC and EF, cut the conic in B_1, C_1; let A_1 be the pole of B_1C_1 and take $A_1B_1C_1$ as the triangle of reference, so that the equation to the conic is

$$\beta\gamma = k\alpha^2 \quad \text{.........................(1).}$$

Let the parameters of the points A, B, C, D, E, F be t_1, t_2, t_3, t_4, t_5, t_6.

The equations of AB and DE are

$$-k(t_1+t_2)\alpha + \beta t_1 t_2 + \gamma k = 0, \qquad \text{(Art. 100)}$$

and

$$-k(t_4+t_5)\alpha + \beta t_4 t_5 + \gamma k = 0.$$

Since they meet on $\alpha = 0$,

$$\therefore\ t_1 t_2 = t_4 t_5 \quad \text{.....................(2).}$$

So, since BC, EF meet on $\alpha = 0$,

$$\therefore\ t_2 t_3 = t_5 t_6 \quad \text{.....................(3).}$$

From (2) and (3), we have

$$t_3 t_4 = t_6 t_1,$$

so that similarly the point of intersection, R, of CD and FA lies on PQ.

Brianchon's Theorem. Draw the tangents at $A, B, C, \ldots$ so as to form a hexagon.

The tangents at A, B meet at the point

$$(t_1 + t_2,\ 2k,\ 2t_1 t_2). \qquad \text{(Art. 100.)}$$

The tangents at D, E meet at

$$(t_4 + t_5,\ 2k,\ 2t_4 t_5).$$

Since $t_4 t_5 = t_1 t_2$, these two points lie on the line

$$\beta t_1 t_2 - k\gamma = 0,$$

which passes through the point A_1.

Similarly the other two lines joining opposite vertices of the new hexagon pass through A_1, and so they are concurrent.

Cor. It follows that the point of concurrence of the three diagonals of a Brianchon hexagon is the pole of the Pascal line of the hexagon formed by joining the points of contact of the sides of the Brianchon hexagon.

Also that the Pascal line of a Pascal hexagon is the polar of the Brianchon point of the hexagon formed by the intersections of the tangents at the angular points of the Pascal hexagon.

109. By means of Pascal's Theorem we can construct a conic passing through five given points A, B, C, D and E. For through E draw any line to meet the conic in a point X to be determined. By Pascal's Theorem the intersections of AB and DE, BC and EX, and CD and XA are collinear. Hence, if the first two pairs meet in P and Q, and PQ meet CD in R, then RA cuts EX in the required point X. Thus, by drawing different lines EX through E, we can determine as many points X as we wish. This construction clearly simplifies into: Draw any line through P the point of intersection of AB and DE and let BC, DC meet this line in Q and R; then the intersection of QE and RA is a point X on the conic.

110. Similarly, by Brianchon's Theorem, we can construct by means of tangents a conic which touches five given straight lines P, Q, R, S, T. For let P and Q meet in the point Y, Q and R in Z, R and S in U, and S and T in V. Take any point *X* on the line *P*; join *XU* and *YV* to meet in *O*, and let *ZO* meet the line *T* in *W*; then *XW* is the other tangent through *X*. By taking different points *X* on the line *P* we can construct as many tangents as we please. This construction clearly simplifies into: Take any point *O* on *YV*; draw *UO* and *ZO* to meet the tangents *P* and *T* respectively in the points *X* and *W*; then *XW* is a tangent.

111. By Pascal's Theorem we can draw the tangent at any point *A* of a conic through *A*, *B*, *C*, *D*, *E*. For, in this case, the point *F* of the general theorem is the next point to *A* on the curve, and indefinitely near coincidence with it. The line *EF* of the general construction is now *EA*. Let then *AB* meet *DE* in *P*, *BC* meet *EF*, *i.e.* *EA*, in *Q*, and *CD* meet *PQ* in *R*; then *R* lies on *FA*, *i.e.* on the tangent at *A*. Hence, joining *RA*, we have the tangent at *A*.

112. By Brianchon's Theorem we can find the point of contact of any tangent *P* of a conic which touches the five lines *P*, *Q*, *R*, *S*, *T*. For let *P* and *Q* meet in *Y*, *Q* and *R* in *Z*, *R* and *S* in *U*, *S* and *T* in *V*, *T* and *P* in *W*, and let *X* be the required point of contact of *P*. We may then consider the one tangent *WY* to be replaced by the two tangents *WX*, *XY* which are indefinitely near coincidence, and whose meeting point is *X*. Then *XYZUVW* is a hexagon circumscribed to a conic so that *XU*, *YV*, and *ZW* meet in a point, *i.e.* *X* is the point where the line joining *U* to the point of intersection of *YV* and *ZW* meets *WY*.

EXAMPLES.

1. From Pascal's theorem deduce that if a conic circumscribes a triangle the tangents at the angular points meet the opposite sides in collinear points.

2. From Pascal's theorem deduce that the tangents to a conic at two opposite vertices of an inscribed quadrilateral meet on the third diagonal.

3. Shew that the hexagon formed by the six lines in order which are obtained by joining alternate pairs of vertices of a Brianchon hexagon *ABCDEF* is a Pascal hexagon. [For the triangles *AEC*, *DBF* are copolar and therefore coaxal.]

Similarly the hexagon formed by producing to meet alternate sides of a Pascal hexagon is a Brianchon hexagon.

4. From any point of the circum-circle of a triangle ABC perpendiculars are drawn to the sides BC, CA, AB which meet the circle again in D, E, F respectively. Shew that a conic can be drawn to touch the six straight lines AB, BC, CD, DE, EF, and FA.

5. From Brianchon's theorem deduce that the ortho-centre of a triangle circumscribing a parabola is on the directrix.

6. If two triangles are in perspective deduce from Pascal's and Brianchon's theorems that the straight lines joining non-corresponding vertices touch a conic, and that the points of intersection of non-corresponding sides lie on a conic.

7. From the theorem that if two triangles are inscribed in a conic they also are circumscribed to a conic, deduce that the 60 Pascal lines formed from six points intersect three by three in 20 points.

Make use of Pascal's Theorem to construct a conic, given:

8. four points and the tangent at one of them [F and A coincide but FA is given in direction].

9. four points and the direction of an asymptote.

10. three points and two lines parallel to the asymptotes, finding also the finite points where the two lines cut the curve.

11. one point and the position of both asymptotes. [The two points at infinity Ω_1 and Ω_1' on one asymptote are given, and so Ω_2 and Ω_2'.]

12. two points, the position of one asymptote, and the direction of the other.

Make use of Brianchon's Theorem to construct a conic, given:

13. four tangents, and the point of contact of one of them, [*i.e.* really five tangents, two being coincident in direction, but with a definite point of intersection].

14. three tangents, and a given line as an asymptote.

15. four tangents, the conic being a parabola. [If two of the tangents meet the line at infinity in Ω, Ω', then $\Omega\Omega'$ is the fifth tangent.]

16. Given four points on a conic, and the tangent at one of the points, construct the tangents at the other three points.

17. Given four tangents to a conic, and the point of contact of one of them, find the points of contact of the other three tangents.

Harmonic Locus and Harmonic Envelope.

113. *To shew that the locus of a point the tangents from which to two given conics form a harmonic pencil [the tangents to each conic being conjugates] is a conic having a common self-conjugate triangle with the given conics.*

Take the common self-conjugate triangle of the two given conics as the triangle of reference, so that the two conics are

$$S \equiv L\alpha^2 + M\beta^2 + N\gamma^2 = 0,$$

and

$$S' \equiv L_1\alpha^2 + M_1\beta^2 + N_1\gamma^2 = 0.$$

The equation to the tangents from $(\alpha', \beta', \gamma')$ to the first conic is

$$(L\alpha'^2 + M\beta'^2 + N\gamma'^2)(L\alpha^2 + M\beta^2 + N\gamma^2) = (L\alpha\alpha' + M\beta\beta' + N\gamma\gamma')^2.$$

The equation to the straight lines joining the point A to the intersection of these tangents with BC is obtained by putting $\alpha = 0$ in this equation, and is thus

$$M\beta^2(L\alpha'^2 + N\gamma'^2) - 2MN\beta'\gamma'\beta\gamma + N\gamma^2(L\alpha'^2 + M\beta'^2) = 0 \quad \text{......(1)}.$$

The similar pair of straight lines for the second conic are

$$M_1\beta^2(L_1\alpha'^2 + N_1\gamma'^2) - 2M_1N_1\beta'\gamma'\beta\gamma + N_1\gamma^2(L_1\alpha'^2 + M_1\beta'^2) = 0 \quad \text{......(2)}.$$

If the four tangents form a harmonic pencil (the tangents to the first conics being conjugates and also the tangents to the second), so also do these two pairs of straight lines, and hence, by Art. 61,

$$MN_1(L\alpha'^2 + N\gamma'^2)(L_1\alpha'^2 + M_1\beta'^2) + M_1N(L_1\alpha'^2 + N_1\gamma'^2)(L\alpha'^2 + M\beta'^2) = 2MNM_1N_1\beta'^2\gamma'^2.$$

Hence the locus of $(\alpha', \beta', \gamma')$ is the conic

$$LL_1(MN_1 + M_1N)\alpha^2 + MM_1(NL_1 + N_1L)\beta^2 + NN_1(LM_1 + L_1M)\gamma^2 = 0,$$

which has the triangle of reference as a self-conjugate conic.

This conic is known as the Harmonic Locus, and is often denoted by $\mathbf{F} = 0$.

114. *To shew that the envelope of a line whose points of intersection with two given conics form a harmonic range (the two points of intersection with each conic being conjugates) is a conic having a common self-conjugate triangle with the given conics.*

The conics being

$$L\alpha^2 + M\beta^2 + N\gamma^2 = 0 \text{ and } L_1\alpha^2 + M_1\beta^2 + N_1\gamma^2 = 0,$$

the equation to the straight lines joining the intersections of the first conic with the straight line

$$l\alpha + m\beta + n\gamma = 0 \quad \dots\dots\dots\dots\dots(1)$$

to the point A of the triangle of reference is

$$L(m\beta + n\gamma)^2 + l^2(M\beta^2 + N\gamma^2) = 0,$$

i.e. $\quad (Lm^2 + Ml^2)\beta^2 + 2Lmn\beta\gamma + (Ln^2 + Nl^2)\gamma^2 = 0.$

So, for the second conic, the corresponding equation is

$$(L_1m^2 + M_1l^2)\beta^2 + 2L_1mn\beta\gamma + (L_1n^2 + N_1l^2)\gamma^2 = 0.$$

These form a harmonic pencil if (Art. 61)

$$(Lm^2 + Ml^2)(L_1n^2 + N_1l^2) + (L_1m^2 + M_1l^2)(Ln^2 + Nl^2) = 2LL_1m^2n^2,$$

i.e. if

$$l^2(MN_1 + M_1N) + m^2(NL_1 + N_1L) + n^2(LM_1 + L_1M) = 0 \quad \dots\dots(2).$$

But (Art. 94) the envelope of the straight line (1) with the condition (2) is

$$\frac{\alpha^2}{MN_1 + M_1N} + \frac{\beta^2}{NL_1 + N_1L} + \frac{\gamma^2}{LM_1 + L_1M} = 0.$$

This conic is known as the Harmonic Envelope, and is often denoted by $\mathbf{F}' = 0$.

115. Similarly, if we proceed, as in the two previous articles, to find the Harmonic Locus and Harmonic Envelope for the conics given by the general equations

$$S \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0,$$

and $\quad S' \equiv a'\alpha^2 + b'\beta^2 + c'\gamma^2 + 2f'\beta\gamma + 2g'\gamma\alpha + 2h'\alpha\beta = 0,$

we should obtain for the harmonic locus

$$\mathbf{F} \equiv (BC' + B'C - 2FF')\alpha^2 + \dots + \dots + 2(GH' + G'H - AF' - A'F)\beta\gamma + \dots + \dots = 0,$$

where A, B, ..., A', B', ... have the same meaning as in Art. 72, and for the tangential equation Φ of the harmonic envelope we should have

$$\Phi \equiv (bc' + b'c - 2ff')\, l^2 + \ldots + \ldots$$
$$+ 2(gh' + g'h - af' - a'f)\, mn + \ldots + \ldots = 0,$$

from which the point equation $\mathbf{F}' = 0$ of the harmonic envelope could be found. The work is however very heavy, and the results are only put here for reference.

Ex. 1. If both S and S' touch a given line at a given point, shew that the harmonic locus and the harmonic envelope touch the given line at the same point.

Ex. 2. When two conics have contact of the third order, the harmonic locus and harmonic envelope coincide.

[Take the angular point C of the triangle of reference as the point of contact, BC as the tangent at C, and CA as the polar of B; then the conics are $a\alpha^2 + b\beta^2 + 2g\gamma\alpha = 0$, and $a'\alpha^2 + b\beta^2 + 2g\gamma\alpha = 0$. The two loci are given by $(a + a')\,\alpha^2 + 2b\beta^2 + 4g\gamma\alpha = 0$, i.e. by $S + S' = 0$.]

Ex. 3. Shew that the harmonic locus and harmonic envelope of the conics $\beta\gamma = k\alpha^2$ and $\beta\gamma = k_1\alpha^2$ are $\beta\gamma = r\alpha^2$ and $\beta\gamma = s\alpha^2$, where r and s are the harmonic and arithmetic means between k and k_1.

Ex. 4. Shew that if $8k_1k_2k_3 = -1$, then for any two of the conics $\beta\gamma = k_1\alpha^2$, $\gamma\alpha = k_2\beta^2$, and $\alpha\beta = k_3\gamma^2$, the harmonic locus and harmonic envelope are the third of the conics. [Three such conics are called a harmonic system.]

116. *Tangential equation to the common points of* $S \equiv L\alpha^2 + M\beta^2 + N\gamma^2 = 0$ *and* $S' \equiv L_1\alpha^2 + M_1\beta^2 + N_1\gamma^2 = 0$.

We want the condition that the straight line

$$l\alpha + m\beta + n\gamma = 0$$

may pass through the intersections of S and S'.

Solving $S = 0$ and $S' = 0$ for α, β, γ we see that the condition is

$$\pm l\sqrt{MN_1 - M_1N} \pm m\sqrt{NL_1 - N_1L} \pm n\sqrt{LM_1 - L_1M} = 0,$$

i.e.

$$l^4(MN_1 - M_1N)^2 + \ldots + \ldots$$
$$- 2m^2n^2(NL_1 - N_1L)(LM_1 - L_1M) - \ldots - \ldots = 0,$$

i.e.

$$[l^2(MN_1+M_1N)+m^2(NL_1+N_1L)+n^2(LM_1+L_1M)]^2$$
$$=4l^4MNM_1N_1+\ldots+\ldots+4m^2n^2LL_1(MN_1+M_1N)+\ldots+\ldots$$
$$=4(l^2MN+m^2NL+n^2LM)(l^2M_1N_1+m^2N_1L_1+n^2L_1M_1),$$

i.e.
$$\Phi^2=4\Sigma\Sigma',$$

where $\Sigma=0$, $\Sigma'=0$ are the tangential equations to the two conics, and $\Phi=0$ is the tangential equation to their harmonic envelope. [Art. 114, equation (2).]

Again the tangents at the common points of S and S' are

$$\pm L\sqrt{(MN_1-M_1N)}\,\alpha\pm M\sqrt{(NL_1-N_1L)}\,\beta$$
$$\pm N\sqrt{(LM_1-L_1M)}\,\gamma=0,$$

and

$$\pm L_1\sqrt{(MN_1-M_1N)}\,\alpha\pm M_1\sqrt{(NL_1-N_1L)}\,\beta$$
$$\pm N_1\sqrt{(LM_1-L_1M)}\,\gamma=0.$$

The coefficients of α, β, γ in each of these equations satisfy the equation

$$\Phi\equiv l^2(MN_1+M_1N)+\ldots+\ldots=0,$$

i.e. the line coordinates of these eight common tangents all satisfy the tangential equation of the harmonic envelope.

Hence *The eight common tangents at the four common points of two conics all touch another conic which is the harmonic envelope of the two conics.*

117. *Equation to the common tangents to the conics* $S\equiv L\alpha^2+M\beta^2+N\gamma^2=0$ *and* $S'\equiv L_1\alpha^2+M_1\beta^2+N_1\gamma^2=0$.

The two conics are touched by $l\alpha+m\beta+n\gamma=0$ if

$$\frac{l^2}{L}+\frac{m^2}{M}+\frac{n^2}{N}=0 \text{ and } \frac{l^2}{L_1}+\frac{m^2}{M_1}+\frac{n^2}{N_1}=0,$$

so that

$$\frac{\pm l}{\sqrt{LL_1(MN_1-M_1N)}}=\frac{\pm m}{\sqrt{MM_1(NL_1-N_1L)}}$$
$$=\frac{\pm n}{\sqrt{NN_1(LM_1-L_1M)}},$$

and the common tangents are therefore

$$\pm\alpha\sqrt{LL_1(MN_1-M_1N)}\pm\beta\sqrt{MM_1(NL_1-N_1L)}$$
$$\pm\gamma\sqrt{NN_1(LM_1-L_1M)}=0,$$

i.e.

$$\alpha^4L^2L_1^2(MN_1-M_1N)^2+\ldots+\ldots$$
$$-2\beta^2\gamma^2MM_1NN_1(NL_1-N_1L)(LM_1-L_1M)-\ldots-\ldots=0,$$

i.e.

$$[\alpha^2LL_1(MN_1+M_1N)+\beta^2MM_1(NL_1+N_1L)$$
$$+\gamma^2NN_1(LM_1+L_1M)]^2$$
$$=4\alpha^4L^2L_1^2MM_1NN_1+\ldots+\ldots$$
$$+4LL_1MM_1NN_1(MN_1+M_1N)\beta^2\gamma^2+\ldots+\ldots$$
$$=4LL_1MM_1NN_1(L\alpha^2+M\beta^2+N\gamma^2)(L_1\alpha^2+M_1\beta^2+N_1\gamma^2),$$

i.e. $$\mathbf{F}^2=4\Delta\Delta' SS',$$

where Δ, Δ' are the discriminants of S and S' and $\mathbf{F}=0$ is the equation to their harmonic locus. [Art. 113.]

Also the points of contact of the common tangents with S are easily seen to be

$$\pm\sqrt{\frac{L_1}{L}(MN_1-M_1N)},\quad\pm\sqrt{\frac{M_1}{M}(NL_1-N_1L)},$$
$$\pm\sqrt{\frac{N_1}{N}(LM_1-L_1M)},$$

and those for S' are

$$\pm\sqrt{\frac{L}{L_1}(MN_1-M_1N)},\quad\pm\sqrt{\frac{M}{M_1}(NL_1-N_1L)},$$
$$\pm\sqrt{\frac{N}{N_1}(LM_1-L_1M)}.$$

These all clearly lie on the conic

$$\mathbf{F}\equiv LL_1(MN_1+M_1N)\alpha^2+MM_1(NL_1+N_1L)\beta^2$$
$$+NN_1(LM_1+L_1M)\gamma^2=0.$$

Hence *The eight points of contact of the common tangents to two conics lie on another conic which is the harmonic locus of the two conics.*

118. *To find the locus of a point the tangents from which to two given conics form a pencil of constant cross-ratio equal to λ.*

With the notation of Art. 113, if the straight lines (1) and (2) are respectively

$$(\beta - p_1\gamma)(\beta - p_3\gamma) = 0, \text{ and } (\beta - p_2\gamma)(\beta - p_4\gamma) = 0,$$

we have
$$\frac{(p_2 - p_1)(p_4 - p_3)}{(p_3 - p_2)(p_1 - p_4)} = \lambda,$$

so that

$$\frac{\lambda - 1}{\lambda + 1} = \frac{(p_1 - p_3)(p_2 - p_4)}{2p_1p_3 + 2p_2p_4 - (p_1 + p_3)(p_2 + p_4)}.$$

Now

$$\begin{aligned}(p_1 - p_3)^2 &= (p_1 + p_3)^2 - 4p_1p_3 \\ &= \frac{4M^2N^2\beta'^2\gamma'^2 - 4MN(L\alpha'^2 + N\gamma'^2)(L\alpha'^2 + M\beta'^2)}{M^2(L\alpha'^2 + N\gamma'^2)^2} \\ &= \frac{-4LMN\alpha'^2 . S}{M^2(L\alpha'^2 + N\gamma'^2)^2},\end{aligned}$$

and similarly for $(p_2 - p_4)^2$.

Hence

$$\frac{\lambda - 1}{\lambda + 1} = \frac{\alpha'^2\sqrt{16LMNS . L_1M_1N_1S'}}{2\alpha'^2\left[LL_1(MN_1 + M_1N)\alpha'^2 + \dots + \dots\right]},$$

after some reduction, *i.e.*

$$\left(\frac{\lambda - 1}{\lambda + 1}\right)^2 \mathbf{F}^2 = 4LMNL_1M_1N_1SS' = 4\Delta\Delta' SS'$$

is the equation to the required locus.

Cor. If $\lambda = -1$, so that the cross-ratio is harmonic, the locus reduces as before to $\mathbf{F} = 0$.

If $\lambda = 1$, then, as in Art. 8, the tangents drawn to one of the conics coincide, and the locus reduces to the conics themselves.

If $\lambda = 0$, the tangent drawn to one of the conics must coincide with one of those drawn to the other conic, and the point from which they are drawn must lie on one of the common tangents. Hence the locus reduces to $\mathbf{F}^2 = 4\Delta\Delta' SS'$, which we have already seen to be the equation to the four common tangents.

Constant Cross-Ratio Property of Conics.

119. *To shew that the cross-ratio of the lines joining any variable point of a conic to four fixed points on it is constant; and that four fixed tangents cut any fifth tangent in a range whose cross-ratio is constant, and equal to that subtended by the points of contact of the four fixed tangents at any point of the conic.*

Take as triangle of reference any two tangents AB, AC to the conic and their chord of contact BC, so that its equation is

$$\beta\gamma = k\alpha^2.$$

Let the fixed points be "t_1," "t_2," "t_3," and "t_4," and let any other point be "t."

The line joining "t" and "t_1" is

$$\alpha k (t + t_1) - \beta t t_1 - \gamma k = 0, \qquad \text{(Art. 100)}$$

i.e.

$$(\gamma - \alpha t) k = t_1 (\alpha k - \beta t).$$

So the other three lines are

$$(\gamma - \alpha t) k = t_2 (\alpha k - \beta t),$$

$$(\gamma - \alpha t) k = t_3 (\alpha k - \beta t),$$

and

$$(\gamma - \alpha t) k = t_4 (\alpha k - \beta t).$$

By Art. 63 their cross-ratio

$$= \frac{(t_2 - t_1)}{(t_3 - t_2)} \frac{(t_4 - t_3)}{(t_1 - t_4)},$$

and is thus the same for all values of "t."

The tangent at "t_1" is

$$2\alpha k t_1 - \beta t_1^2 - \gamma k = 0,$$

and that at "t" is

$$2\alpha k t - \beta t^2 - \gamma k = 0.$$

Hence the line joining their point of intersection to the angular point C of the triangle of reference is

$$2\alpha k (t_1 - t) = \beta (t_1^2 - t^2),$$

i.e.

$$2\alpha k - \beta t = t_1 \beta.$$

The corresponding lines for the other three tangents are

$$2\alpha k - \beta t = t_2 \beta,\ 2\alpha k - \beta t = t_3 \beta, \text{ and } 2\alpha k - \beta t = t_4 \beta.$$

By Art. 63 their cross-ratio

$$=\frac{(t_2-t_1)(t_4-t_3)}{(t_3-t_2)(t_1-t_4)},$$

and this by the foregoing is the cross-ratio of the pencil subtended at any point of the conic by the points of contact of the four fixed tangents.

Cor. If we take two of the fixed points as the points B, C of the triangle of reference then $t_1=0$, $t_3=\infty$, and the cross-ratio of the pencil subtended becomes $\frac{t_2}{t_4}$.

Ex. Deduce Pascal's Theorem from the cross-ratio property of four points on a conic.

[With the notation of Art. 105 let AB meet CD in S and BC meet DE in T. Then $A(CDFB)=E(CDFB)$. Hence, considering the transversals CD and CB of the pencils, $(CDRS)=(CTQB)$. $\therefore P(CDRS)=P(CTQB)$, *i.e.* $P(CDRB)=P(CDQB)$. Hence PR and PQ must be the same ray, so that PQR is a straight line.]

EXAMPLES.

1. If the pencil subtended by four points B, P, C, Q at any point of the conic passing through them is harmonic, shew that BC and PQ are conjugate with regard to the conic. [If the triangle of reference is BC and the tangents AB, AC at B and C, then, by Cor. Art. 119, P is the point "t_2" and Q the point "$-t_2$." The equation to the line through them then becomes $\beta t_2^2=k\gamma$, which passes through A, the pole of BC. Hence etc.]

2. From the focus and directrix property of a conic deduce the constant cross-ratio property of four points on a conic.

[If A, B, C, D are the fixed points and the lines joining them to any point P meet the directrix in a, b, c, d, then

$$\angle bSa=\angle aSP-\angle bSP=90^\circ-\tfrac{1}{2}\angle ASP-90^\circ+\tfrac{1}{2}\angle BSP=\tfrac{1}{2}\angle ASB=\frac{a}{2}.$$

$$\therefore P(ABCD)=(abcd)=S(abcd)=\text{etc.}]$$

3. By taking two of the four fixed points in the theorem of Art. 119 as the circular points at infinity, deduce that the angle in any segment of a circle is constant.

4. Shew that the locus of a point P, such that the straight lines joining it to the angular points of the triangle of reference and to a fixed point $(\alpha', \beta', \gamma')$ form a pencil of constant cross-ratio equal to $-k$, is

$$\beta\gamma\alpha'-\gamma\alpha(k+1)\beta'+\alpha\beta k\gamma'=0.$$

5. If B, P, C, Q are four points on a conic, and A is the pole of BC, shew that the cross-ratio of the pencil $A\,(BPCQ)$ is the square of the pencil subtended by B, P, C, Q at any point of the conic. [Take ABC as the triangle of reference.]

6. Shew that one of the cross-ratios of the pencil joining any point of the conic

$$S+\lambda S' \equiv ax^2+by^2+cz^2+\lambda\,(a'x^2+b'y^2+c'z^2)=0$$

to the four points given by $S=0$ and $S'=0$ is $\dfrac{b+\lambda b'}{c+\lambda c'} \times \dfrac{ac'-a'c}{ab'-a'b}$, and explain the result when λ has either of the values $-\dfrac{a}{a'}$, $-\dfrac{b}{b'}$ or $-\dfrac{c}{c'}$.

7. The four sides PQ, QR, RS, SP of a quadrilateral touch a conic, and p, q, r, s are the perpendiculars from P, Q, R, S upon any tangent to the conic; shew that $pr=kqs$, where k is a constant.

[Use the second part of Art. 119.]

8. Apply the harmonic property of four points on a conic to shew that the intercept made by the asymptotes of a hyperbola on any tangent is bisected at the point of contact.

[We have $\Omega\,(PQ\Omega\Omega')=\Omega'\,(PQ\Omega\Omega')$, where P, Q are any two points on the curve and Ω, Ω' its two points at infinity. Replace these cross-ratios for pencils by the cross-ratios for the corresponding ranges on the two asymptotes.]

Homographic Ranges on Conic Sections.

120. If on a conic we have two ranges of points

$$P, Q, R, S, \ldots \quad \text{and} \quad P', Q', R', S', \ldots,$$

such that the cross-ratio of the pencil subtended at any point O of the conic by any four points of the first range is equal to the cross-ratio of the pencil subtended at O by the corresponding four points of the second range, then the two ranges of points are said to be **Homographic**.

Taking the notation of Art. 119, we have shewn that if P, Q, R, S be the points t_1, t_2, t_3, t_4 and P', Q', R', S' the points t_1', t_2', t_3', t_4', the cross-ratio of the pencil subtended by these two sets of four points at any point of the conic are the same if

$$\frac{(t_2-t_1)\,(t_4-t_3)}{(t_3-t_2)\,(t_1-t_4)} \equiv \frac{(t_2'-t_1')\,(t_4'-t_3')}{(t_3'-t_2')\,(t_1'-t_4')},$$

i.e. if

$$Kt_4t_4'+Lt_4+Mt_4'+N=0 \quad \ldots\ldots\ldots\ldots(1),$$

where K, L, M, N contain only t_1, t_2, t_3, t_1', t_2', t_3' and not t_4 or t_4'.

Hence, if S and S' are any two points of the homographic ranges given by P, Q, R and P', Q', R' the relation between their parameters must be given by an equation of the form (1).

Just as in Art. 17, the equation (1) is the most general rational algebraic relation between the parameters of S and S' to ensure that to each point S of the first range there is one, and one only, point S' of the second range, and that to each point S' of the second range there is one, and one only, point S of the first range, *i.e.* to ensure that there is a one-one correspondence.

It is easy to prove, by help of Art. 119, that, as in Art. 18, two ranges of points on the conic $\beta\gamma = k\alpha^2$, the parameters of whose corresponding points are connected by an equation of the form

$$Ktt' + Lt + Mt' + N = 0 \quad \text{................(2)}$$

are homographic.

There will be double points on the two ranges, *i.e.* points each of which corresponds to itself, if $t' = t$, and hence the double points are given by the equation

$$Kt^2 + (L + M)t + N = 0 \quad \text{...............(3).}$$

Hence there are two double points, which are real, coincident, or imaginary.

121. *Simplification when the double points are real.*

In this case take the points B and C of the triangle of reference as the two double points. As in Art. 101 we have seen that, for B, $t = 0$, and, for C, $t = \infty$. The equation (3) is then satisfied by the two values $t = 0$ and $t = \infty$, so that $K = 0$ and $N = 0$. The relation (2) then becomes

$$t' = -\frac{L}{M}t = \lambda t,$$

where λ is a constant.

Hence in the conic $\beta\gamma = k\alpha^2$ the relation between the parameters of the corresponding points of two homographic ranges, whose double points are the angular points, B and C, of the triangle of reference, reduces to the form $t' = \lambda t$.

122. *Two homographic ranges $P, Q, R, \ldots$ and $P', Q', R', \ldots$ are taken on a conic; the locus of the points of intersection of all such pairs as PQ' and $P'Q$, joining two pairs of corresponding points, is the straight line joining the double points of the two ranges, and the envelope of the line joining any two corresponding points, such as P and P', of the ranges is a conic having double contact with the given one.*

Let B and C be the double points of the ranges, and let the tangents at B and C meet in A. With ABC as the triangle of reference, the equation to the conic is

$$\beta\gamma = k\alpha^2.$$

As in the last article the relations between the parameters of corresponding points of the two ranges is now $t' = \lambda t$, where λ is some constant.

If P, Q, R, ... are the points "t_1," "t_2," "t_3," ... and P', Q', R' the points "t_1'," "t_2'," "t_3'," ... etc., the equation to PQ' is

$$\alpha k(t_1 + t_2') - \beta t_1 t_2' - \gamma k = 0,$$

i.e.

$$\alpha k(t_1 + \lambda t_2) - \beta\lambda t_1 t_2 - \gamma k = 0.$$

So that of $P'Q$ is

$$\alpha k(t_2 + \lambda t_1) - \beta\lambda t_1 t_2 - \gamma k = 0.$$

These clearly meet on the line $\alpha = 0$, which is the line joining the double points. It is called the **homographic or cross-axis** of the homography.

Again the equation to PP' is

$$\alpha k(t_1 + t_1') - \beta t_1 t_1' - \gamma k = 0,$$

i.e.

$$\alpha k(\lambda + 1) t_1 - \beta\lambda t_1^2 - \gamma k = 0,$$

the envelope of which is $\beta\gamma = \dfrac{k(1+\lambda)^2}{4\lambda}\alpha^2$, a conic having double contact with the given one at the double points.

123. Without using the double points of the ranges (which are often imaginary) as two of the angular points of the triangle of reference, we can somewhat simplify the work by taking one point of the first range as B and the corresponding point as C.

The general relation $Ktt' + Lt + Mt' + N = 0$ is then satisfied by $t=0$ (for the point B) and $t'=\infty$ (for the point C). Hence in this case $M=0$, and the general relation connecting pairs of corresponding points becomes

$$Ktt' + Lt + N = 0 \quad \text{..................(1)}.$$

If P, Q, R, ... of the first range are the points "t_1," "t_2," "t_3," ..., and P', Q', R', ..., the corresponding points of the second range, are the points "t_1'," "t_2'," "t_3'," ..., the equation of PQ' is

$$-k\alpha(t_1 + t_2') + \beta t_1 t_2' + \gamma k = 0,$$

i.e., by relation (1),

$$k\alpha[Kt_1t_2 - Lt_2 - N] + \beta(Lt_1t_2 + Nt_1) - \gamma kKt_2 = 0.$$

So the equation to $P'Q$ is

$$k\alpha[Kt_1t_2 - Lt_1 - N] + \beta[Lt_1t_2 + Nt_2] - \gamma kKt_1 = 0.$$

On subtraction, we see that they meet on the line

$$kL\alpha + N\beta + kK\gamma = 0 \quad \text{..................(2)}.$$

Now the double points are from (1) given by

$$Kt^2 + Lt + N = 0,$$

and the equation of the line joining them is easily seen to be (2).

The required locus of cross intersections is therefore, as before, seen to be the line joining the double points (real or imaginary) of the two ranges.

Again the equation of PP' is

$$-k\alpha(t_1 + t_1') + \beta t_1 t_1' + \gamma k = 0,$$

i.e. from (1)

$$t_1^2(kK\alpha + L\beta) - t_1(kL\alpha - N\beta + kK\gamma) - kN\alpha = 0,$$

and its envelope is therefore

$$(kL\alpha - N\beta + kK\gamma)^2 = -4kN\alpha(kK\alpha + L\beta),$$

i.e. $$(kL\alpha + N\beta + kK\gamma)^2 = 4kKN(\beta\gamma - k\alpha^2),$$

i.e. a conic having double contact with the given one at the (real or imaginary) double points of the ranges.

124. If P, Q, R, ... and P', Q', R', ... are the two sets of homographic points, we have shewn that the intersections of PQ' and $P'Q$, QR' and $Q'R$, RP' and $R'P$ lie on a straight line. But this straight line is the Pascal line of the hexagon $PQ'RP'QR'$. Hence the double points of the system of homographic points lie on the Pascal line of one of the hexagons which is formed by three points of one range and the three corresponding points on the other range.

125. *Construction of homographic ranges on a conic.*

Just as in Art. 22, two sets of three points P, Q, R and P', Q', R', all lying on the conic, are required to determine two homographic ranges. When these are given we can easily determine as many points as we please. For join PQ' and $P'Q$ to meet in U, and QR' and $Q'R$ to meet in V. Then UV is the homographic, or cross-, axis of the two ranges. If UV meets the conic in real points, they are the double points of the ranges. If UV does not meet the conic in real points, the double points are imaginary.

To construct the point on the second range corresponding to any point X on the conic belonging to the first range. Join $P'X$ to meet the homographic axis UV in W and let PW meet the conic in X'. Then X' is the required point which corresponds to X. So any number of pairs of points, such as X and X', may be found.

126. *Construction of double points on co-axal homographic ranges.*

Let the co-axal ranges be $A, B, C, \ldots, A', B', C', \ldots$. Draw any convenient circle, or conic, and join any point K of this circle to the above points meeting the circle in $P, Q, R, \ldots, P', Q', R', \ldots$. Construct the homographic axis as before, and let it meet the circle in O and O', so that KO and KO' are the double rays of the pencils subtended by the ranges $P, Q, R, \ldots, P', Q', R', \ldots$. Then KO and KO' produced meet the original axis of the ranges $A, B, C, \ldots$ and $A', B', C', \ldots$ in the required double points of these two ranges.

If we construct X, X' as in the last article then KX, KX' will meet the original axis in corresponding points of the two homographic ranges.

127. *Two homographic sets of tangents $p, q, r, \ldots$ and $p', q', r', \ldots$ are taken for a conic; the line joining the intersection of p and q' to the intersection of p' and q passes* *through the point of intersection of the double tangents, and the locus of the point of intersection of any two corresponding tangents, p and p′, is a conic having double contact with the given one.*

Take the double tangents as the sides AB, AC of the triangle of reference and their points of contact as B and C. Then as in Art. 121 the parameters of the points of contact of two corresponding tangents are connected by a relation $t' = \lambda t$.

If the points of contact of the tangents $p, q, r, \ldots$ are "t_1," "t_2," "t_3," ... and those of $p', q', r', \ldots$ are "t_1'," "t_2'," "t_3'," ... the equation of the tangent p is

$$-2akt_1 + \beta t_1^2 + \gamma k = 0,$$

and that of the tangent q' is

$$-2akt_2' + \beta t_2'^2 + \gamma k = 0.$$

Their intersection is given by

$$\frac{\alpha}{t_1 + t_2'} = \frac{\beta}{2k} = \frac{\gamma}{2t_1 t_2'},$$

i.e. by

$$\frac{\alpha}{t_1 + \lambda t_2} = \frac{\beta}{2k} = \frac{\gamma}{2\lambda t_1 t_2}.$$

So the intersection of p' and q is given by

$$\frac{\alpha}{t_2 + \lambda t_1} = \frac{\beta}{2k} = \frac{\gamma}{2\lambda t_1 t_2}.$$

These intersections lie on $\dfrac{\beta}{k} = \dfrac{\gamma}{\lambda t_1 t_2}$, which passes through A, the point of intersection of the double tangents.

Again the tangents at P and P' meet where

$$\frac{\alpha}{t_1 + t_1'} = \frac{\beta}{2k} = \frac{\gamma}{2t_1 t_1'},$$

i.e. where

$$\frac{\alpha}{(\lambda + 1)\, t_1} = \frac{\beta}{2k} = \frac{\gamma}{2\lambda t_1^2},$$

so that they meet on the conic $\beta\gamma = \dfrac{4k\lambda}{(\lambda + 1)^2}\alpha^2$, a conic having double contact with the given one at the points of contact of the double tangents.

128. Maclaurin's Theorem. *If the three sides of a triangle pass through fixed points, and the ends of its base move on two fixed straight lines, then the locus of its vertex is a conic.*

Let the two straight lines be OL and OM, and the three fixed points A, B, and C (see the figure on the next page). Through A draw a straight line QAR to meet OL, OM in Q, R; let QB, RC meet in P. We want the locus of P.

Different positions of P being P_1, P_2, P_3, P_4, we have

$$B(P_1P_2P_3P_4) = (Q_1Q_2Q_3Q_4) = A(Q_1Q_2Q_3Q_4) = (R_1R_2R_3R_4) = C(P_1P_2P_3P_4),$$

so that BP and CP give corresponding rays of homographic pencils.

Therefore, as in Art. 25, the locus of P is a conic passing through B and C. It also passes through O; for one position of the line RAQ is when the latter passes through O; in this case R, Q coincide at O and the corresponding lines QB, RC meet at O, and therefore O is one point on the locus. Again, by taking AC as one position of the line through A, we see that the locus passes through the intersection of AC and OL; and similarly through the intersection of AB and OM.

Cor. For the fixed point A could be substituted a conic, touching the two fixed lines, which the base QR must touch.

129. An analytical proof is quite easy; for let the triangle ABC be the triangle of reference and the equations to OL, OM be $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $l_2\alpha + m_2\beta + n_2\gamma = 0$. If the equation to QAR is $\beta = \lambda\gamma$, the equations to BQ and CR are $l_1\alpha + m_1\lambda\gamma + n_1\gamma = 0$ and $\lambda(l_2\alpha + m_2\beta) + n_2\beta = 0$.

Eliminating λ, we have for the locus of P

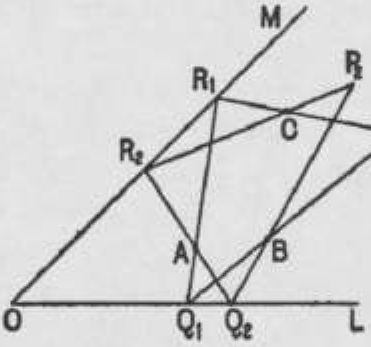
$$(l_1\alpha + n_1\gamma)(l_2\alpha + m_2\beta) = m_1 n_2 \beta\gamma,$$

a conic which clearly passes through B, C and O. It also passes through the intersection of OL and AC, and through the intersection of OM and AB.

130. From Maclaurin's theorem we have an easy way of drawing a conic through five points O, B, C, P_1, P_2.

Through O draw any two convenient lines OL and OM. Let P_1B and P_1C meet OL, OM in Q_1 and R_1; also let P_2B

and P_2C meet them in Q_2 and R_2; draw Q_1R_1 and Q_2R_2 meeting in A.



Then, by Maclaurin's theorem, any triangle with its sides passing through A, B, C and its base angles moving on OL, OM will have for the locus of its vertex a conic through B, C, P_1, P_2, and O. To draw it take *any* point Q on OL; let QA meet OM in R; then the point of intersection of QB and RC will give a point P on the conic required. By taking successive points Q, we can obtain as many points P as we wish.

The student will see after a little practice that many of these lines need not be drawn. For each point Q he can by means of a ruler through A mark the corresponding point R, and then by means of placing the ruler consecutively through R, C and Q, B he need only make two short pencil marks crossing at P.

If it were found to be convenient for the drawing, the lines OL, OM could be taken to pass through P_1 and P_2.

131. Newton's theorem for generating conics. *If two angles of given magnitude turn about fixed vertices, O_1 and O_2, and if two of their legs intersect on a fixed straight line, the locus of the intersection of their other two legs is a conic section passing through the two fixed vertices.*

Let O_1P, O_2P be the legs which intersect in a point P, lying on the fixed straight line; and O_1Q, O_2Q the other legs. If P_1, P_2, P_3, P_4 are different positions of P, then, since $P_1O_1Q_1$, $P_1O_2Q_1$ etc. are constant angles, we have

$$\begin{aligned} O_1(Q_1Q_2Q_3Q_4) &= O_1(P_1P_2P_3P_4) = (P_1P_2P_3P_4) \\ &= O_2(P_1P_2P_3P_4) = O_2(Q_1Q_2Q_3Q_4). \end{aligned}$$

O_1Q and O_2Q are thus corresponding rays of homographic pencils so that, by Art. 25, the locus of Q is a conic passing through O_1 and O_2.

Cor. If the point P, the intersection of the legs, lies on a conic which passes through O_1 and O_2, there is still a one-one correspondence between O_1Q and O_2Q, and the locus of Q is another conic through O_1 and O_2.

132. Ex. 1. *Inscribe a triangle in a given conic whose sides pass through three given points K, L, and M.*

Take any point P_1 on the conic; join P_1K and let it meet the conic in Q_1; join Q_1L and let it meet the conic in R_1; join R_1M and let it meet the conic in P_1'. In general P_1' will not coincide with P_1; if it does the problem is solved; if it does not, take two other points P_2, P_3 on the conic, perform the same constructions as in the case of P_1 and obtain the corresponding points P_2', P_3'. Then P_1 and P_1' have an algebraic one-one correspondence and so have P_2, P_2' and also P_3, P_3'. Hence P_1, P_2, P_3 and P_1', P_2', P_3' determine two homographic ranges. Find their homographic axis by the construction of Art. 122 and let it meet the conic in O_1 and O_2. If O_1 and O_2 are real, we shall obtain the triangle required, by starting with either of them and performing the construction above.

The method here used is known as that of **False Position.**

Ex. 2. *If a triangle be inscribed in a given conic, two of whose sides pass through given points K and L, find the envelope of the third side.*

As in the last example the ranges $P_1, P_2, P_3, \ldots$ and the ranges $R_1, R_2, R_3, \ldots$ are homographic; hence, by Art. 122, the envelope of the third side P_1R_1 is a conic having double contact with the given conic at the double points of the homography determined by P_1, P_2, P_3 and R_1, R_2, R_3 and these double points will be where KL meets the conic.

Ex. 3. *If a polygon be inscribed in a conic and if all its sides but one pass through given points, the envelope of that one side will be a conic having double contact with the given one.*

Let $P_1 P_2 \ldots P_n$ be one such polygon. If P_1 be taken arbitrarily the line joining it to the first fixed point gives P_2 uniquely; the line joining P_2 to the second fixed point gives P_3 uniquely, and so on. Thus P_n is uniquely determined, by an algebraic relation, when P_1 is given. The different positions of P_1, viz. $Q_1, R_1, \ldots$, and the corresponding positions of P_n, viz. $Q_n, R_n, \ldots$, therefore determine homographic ranges, and, by Art. 122, the envelope of the line joining corresponding points, *i.e.* the envelope of the remaining side of the polygon, is a conic having double contact with the given one at the double points of the homography.

EXAMPLES.

1. Two series of rays connecting two fixed points on a conic to a variable point of a conic are homographic.

2. If through a fixed point A are drawn lines AP_1P_2, AQ_1Q_2, ... to meet a conic in P_1, P_2, Q_1, Q_2, ..., then the range $P_1Q_1R_1$... is always homographic with the range $P_2Q_2R_2$....

3. Two corresponding points of homographic ranges on a conic, together with the double points of the ranges, subtend at any point of the conic a pencil of constant cross-ratio.

4. The pole of the line joining the double points of a homography on a conic is joined to any pair of corresponding points of the homography; shew that these lines and the lines from the pole to the double points form a pencil of constant cross-ratio.

5. Two fixed points O, O' are joined to any point Q on a conic and meet the conic again in P and P'; shew that the ranges described by P and P' are homographic, and that their double points are at the intersections of OO' with the conic.

6. Two conics touch each other at B and C; the straight lines joining B, C to any point of one conic meet the other conic in P, Q; shew that the envelope of PQ is a conic having double contact with the two given conics at B and C.

7. Inscribe in a triangle a second triangle whose sides pass through three given points.

8. Inscribe in a conic a polygon all of whose sides pass through given points. [This is an extension of Art. 132, Ex. 1.]

9. Prove the correlative of Maclaurin's Theorem, viz. If the three vertices P, Q, and R of a triangle move on fixed straight lines BC, CA and AB, and if two of its sides, PQ and PR, pass through two fixed points S and T, then the envelope of the third side is a conic. Shew also that this conic touches the straight lines AB, AC, BS, CT and ST. [This theorem can be extended to any polygon.]

10. Circumscribe a triangle to a given conic, so that its vertices may lie on three given straight lines.

11. If a polygon be circumscribed to a conic, and all the angular points except one lie on given straight lines, the locus of that angular point is a conic.

12. The sides of a polygon pass through fixed points, and all the vertices but one lie on a fixed conic; shew that the locus of the remaining vertex is a conic passing through two of the fixed points.

13. If the sides of a polygon pass through fixed points, and all the angular points but one lie on fixed straight lines, shew that the locus of that angular point is a conic.

Involution ranges on a Conic.

133. Involution ranges on a conic are homographic ranges, such that to every point P corresponds the same point P', whether P belongs to the first range or the second. Just as in Art. 28, we see that this can only be the case if in the fundamental relation of Art. 120 we have $L = M$.

With the equation of the conic in the form $\beta\gamma = k\alpha^2$, the parameters of any two mates in an involution are therefore given by

$$Ktt' + L(t + t') + N = 0.$$

The double points are given by

$$Kt^2 + 2Lt + N = 0.$$

In the case when the double points are real and are taken as the angular points B and C of the triangle of reference, we see, as in Art. 121, that $K = 0$ and $N = 0$, and the relation between t and t' becomes $t' = -t$.

Or again, if we take the points B and C of the triangle of reference as two mates in the involution range, the fundamental relation is satisfied by $t = 0$ and $t' = \infty$. Hence in this case $L = 0$, and the relation reduces to

$$tt' = -\frac{N}{K} = \lambda.$$

134. *If P and P', Q and Q', R and R', ... are mates in an involution range on a conic, the locus of the intersection of all such pairs as PQ' and $P'Q$ is the straight line joining the double points of the involution; and the lines joining pairs of mates, such as PP', QQ', ..., are concurrent in a point which is the pole of the line joining the double points of the involution.*

Take B and C as two mates in the involution so that, as in the last article, the fundamental relation is $tt' = \lambda$.

If P, Q, R, ... are the points "t_1," "t_2," "t_3," ... and P', Q', R', ... the points "t_1'," "t_2'," "t_3'," ..., the equation to PQ' is

$$-\alpha k(t_1 + t_2') + \beta t_1 t_2' + \gamma k = 0,$$

i.e.

$$-\alpha k(t_1 t_2 + \lambda) + \beta t_1 \lambda + \gamma k t_2 = 0,$$

and that to $P'Q$ is

$$-ak(t_1t_2+\lambda)+\beta t_2\lambda+\gamma kt_1=0.$$

By subtraction, we see that these meet on the straight line

$$\beta\lambda-\gamma k=0.$$

Now the parameters of the double points are $\sqrt{\lambda}$ and $-\sqrt{\lambda}$, and the equation of the line joining them is thus

$$\beta\lambda-\gamma k=0,$$

so that PQ' and $P'Q$ meet on the line joining the double points. This line is called the **Axis of the Involution.**

Again the equation to PP' is

$$-ak(t_1+t_1')+\beta t_1t_1'+\gamma k=0,$$

i.e.

$$-ak(t_1^2+\lambda)+\beta\lambda t_1+\gamma kt_1=0.$$

This clearly passes through the point $(0, -k, \lambda)$ which is the pole of the line joining the double points. This point is called the **Pole of the Involution.**

Again the equation to PQ is

$$-ak(t_1+t_2)+\beta t_1t_2+\gamma k=0,$$

and that to $P'Q'$ is

$$-ak(t_1'+t_2')+\beta t_1't_2'+\gamma k=0,$$

i.e.

$$-ak\lambda(t_1+t_2)+\beta\lambda^2+\gamma kt_1t_2=0.$$

These clearly meet on the line $\beta\lambda-\gamma k=0$, *i.e.* the Axis of Involution.

Hence the Axis of Involution is easily found when any two pairs of mates P and P', Q and Q', are given; for it passes through the meet of PQ' and $P'Q$, and through the meet of PQ and $P'Q'$.

135. Conversely, *If from any point A we draw straight lines to meet the conic in P and P', Q and Q', R and R', ... we determine an involution range of which the points of contact of the tangents from A to the conic are the (real or imaginary) double points.*

Also given two pairs of points of an involution range on a conic, P and P', Q and Q', we can easily determine as many more points of the range as we wish. For let PP' meet QQ' in A, so that A is the pole of the involution; then the mate of any other point R on the conic is the second point R' in which AR meets the conic.

136. If we assume that the double points of the involution are real, and take them for the points B and C of the triangle of reference, the work is simplified.

For P and P' are now the points "t_1" and "$-t_1$" and Q, Q' are "t_2" and "$-t_2$." The equations to PQ' and $P'Q$ are

$$-ka(t_1-t_2)-\beta t_1 t_2+\gamma k=0 \text{ and } -ka(-t_1+t_2)-\beta t_1 t_2+\gamma k=0,$$

which clearly meet on $a=0$, *i.e.* on BC.

The equation to PP' is $\beta t_1^2-\gamma k=0$, which passes through A. Also the equations to PQ and $P'Q'$ are

$$-ka(t_1+t_2)+\beta t_1 t_2+\gamma k=0, \text{ and } ka(t_1+t_2)+\beta t_1 t_2+\gamma k=0,$$

which also meet on BC.

EXAMPLES.

1. A system of parallel lines meets a conic in pairs of points in involution. [In Art. 135 take the point A at infinity.]

2. The double points of an involution range on a conic, and any pairs of mates in the involution, subtend a harmonic pencil at any point on the conic.

3. The points of contact of pairs of tangents to a conic from points lying on a given straight line determine an involution range on the conic, and the tangents themselves determine an involution range on any fixed tangent to the conic.

[The chords of contact pass through a fixed point, viz. the pole of the given line; hence, by Art. 135, they determine an involution range on the conic.]

4. If O is a point on a conic, and chords PP', QQ', ... subtend right angles at O, the chords pass through a fixed point on the normal at O. [The Frégier Point; see Ex. Art. 404, Part I.]

[OP and OP', OQ and OQ', ... are a pencil in involution and hence determine an involution range on the conic. Hence (Art. 134) PP', QQ', ... meet in a point O'. If P coincides with O, then P' is at the other end of the normal through O, and hence O' must lie on the normal.]

5. If, in the previous example, OP and OP' etc. are equally inclined to the normal at O, the chords pass through a fixed point on the tangent at O.

6. P and P' are fixed points on a given conic; from P and P' pairs of tangents are drawn to any confocal conic and they cut the given conic in Q, R and Q', R'; shew that the locus of the intersection of QR and $Q'R'$ is a conic. [Use the previous example and Art. 25.]

Homographic plane figures.

137. Two plane figures, in different planes or the same plane, are said to be homographic if to each point of either figure there corresponds one, and one only, point of the other figure, and if, in addition, to points on a straight line in the first figure correspond points on a straight line in the second figure. The relation between the two figures is called an homography.

Let (x, y) be the coordinates of a point of the first figure referred to any axes in the first plane, and (x', y') the coordinates of the corresponding point in the second figure. Then if there is a one-one correspondence the most general rational algebraic relation between them is of the form

$$x' = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad y' = \frac{a_2x + b_2y + c_2}{a_4x + b_4y + c_4}.$$

If to points on a straight line are to correspond points on a straight line then, corresponding to a relation of the form $lx' + my' + n = 0$, we must have a linear relation between x and y. But, on substituting the above values, we do not in general obtain a linear relation unless the denominators in the values of x', y' are the same, or one is a constant multiplier of the other. In either case we have

$$x' = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3} \text{ and } y' = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3} \quad \ldots(1).$$

These values, on being substituted in the relation

$$lx' + my' + n = 0,$$

give a linear relation between x and y.

On solving for x and y, we obtain

$$x = \frac{x'(b_2c_3 - b_3c_2) + y'(b_3c_1 - b_1c_3) + (b_1c_2 - b_2c_1)}{x'(a_2b_3 - a_3b_2) + y'(a_3b_1 - a_1b_3) + (a_1b_2 - a_2b_1)},$$

and

$$y = \frac{x'(c_2a_3 - c_3a_2) + y'(c_3a_1 - c_1a_3) + (c_1a_2 - c_2a_1)}{x'(a_2b_3 - a_3b_2) + y'(a_3b_1 - a_1b_3) + (a_1b_2 - a_2b_1)}.$$

The transformation from the second figure to the first is therefore by relations of the same type as that from the first figure to the second.

If $A_1, B_1, C_1, \ldots$ are the co-factors of $a_1, b_1, c_1, \ldots$ in the determinant

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$$

then $x = \dfrac{A_1x' + A_2y' + A_3}{C_1x' + C_2y' + C_3}$ and $y = \dfrac{B_1x' + B_2y' + B_3}{C_1x' + C_2y' + C_3}$.

The vanishing line in the first figure, *i.e.* the line corresponding to the line at infinity in the second figure, is

$$a_3x + b_3y + c_3 = 0$$

and the vanishing line in the second figure is

$$x'(a_2b_3 - a_3b_2) + y'(a_3b_1 - a_1b_3) + (a_1b_2 - a_2b_1) = 0.$$

138. Any four points of the first figure, no three of which are in the same straight line, and four similar points of the second figure determine an homology of the above kind. For we should then have eight linear relations to determine the eight ratios between the nine quantities

$$a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3.$$

It can be shewn that any four coplanar points can be projected into any four coplanar points in any other plane, provided that no three points of either set lie on a straight line. (See Hatton's *Projective Geometry*, Page 43, Russell's *Pure Geometry*, Page 116, or Cremona's *Projective Geometry*, Page 80.)

Coplanar homographic correspondence.

139. If the two figures are in the same plane, we have the same relations between (x, y) and (x', y') as in the previous article. In this case some points of one figure may correspond to themselves in the other figure. These points are found by putting $x' = x$ and $y = y'$ in relation (1). We then obtain

$$x(a_3x + b_3y + c_3) = a_1x + b_1y + c_1 \;\ldots\ldots\ldots(1),$$

and

$$y(a_3x + b_3y + c_3) = a_2x + b_2y + c_2 \;\ldots\ldots\ldots(2).$$

On eliminating y, we have a cubic for x; for each root of this cubic the equation (1) gives one, and one only, value of y. We thus have in general three double points.

The two conics (1), (2) only intersect in three finite points, since they have the direction of one asymptote common, and hence have one point at infinity common.

Ex. Two homographic coplanar figures are such that to the points (0, 0), (0, 1), (1, 0) and (1, 1) correspond the points (0, 0), (0, −1), (2, 1), and (3, 1) respectively. Find the general relation connecting the two sets of points, and shew that the coordinates of the self-corresponding points are (0, 0), (0, 3) and (7, 3). Shew also that the vanishing lines of the two figures are $x-y+2=0$ and $x-3y+3=0$.

Parametric Coordinates.

140. We have had examples of the coordinates of a point on a conic being represented by functions of one independent parameter. Some more general propositions follow.

141. *If the coordinates* (x, y, z), *whether Cartesian, Trilinear or Areal, of a point are given by the relations*

$$\frac{x}{a_1t^2+2b_1t+c_1}=\frac{y}{a_2t^2+2b_2t+c_2}=\frac{z}{a_3t^2+2b_3t+c_3},$$

to shew that its locus is a conic section.

The straight line $lx+my+nz=0$ meets the locus of the point where

$$t^2(a_1l+a_2m+a_3n)+2t(b_1l+b_2m+b_3n)+(c_1l+c_2m+c_3n)=0.$$

This gives, in general, two values of t. The straight line therefore meets the locus in two points, and hence the locus is a conic.

The straight line will touch the curve if this equation has equal roots, *i.e.* if

$$(b_1l+b_2m+b_3n)^2=(a_1l+a_2m+a_3n)(c_1l+c_2m+c_3n),$$

i.e. if $(b_1^2-c_1a_1)l^2+(b_2^2-c_2a_2)m^2+(b_3^2-c_3a_3)n^2$

$$+mn(2b_2b_3-a_2c_3-a_3c_2)+nl(2b_3b_1-a_3c_1-a_1c_3)$$
$$+lm(2b_1b_2-a_1c_2-a_2c_1)=0.$$

This is the tangential equation to the conic. The point equation may be found as in Art. 73. It is

$$x^2(B_1^2-4C_1A_1)+y^2(B_2^2-4C_2A_2)+z^2(B_3^2-4C_3A_3)$$
$$+2yz(B_2B_3-2A_2C_3-2A_3C_2)+2zx(B_3B_1-2A_3C_1-2A_1C_3)$$
$$+2xy(B_1B_2-2A_1C_2-2A_2C_1)=0,$$

where $A_1, B_1, C_1, \ldots$ are the co-factors of $a_1, b_1, c_1, \ldots$ in the determinant

$$\Delta = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}.$$

Or it may be found as follows; from the given equations we have

$$a_1t^2 + 2b_1t + c_1 = \lambda x \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$a_2t^2 + 2b_2t + c_2 = \lambda y \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and $$a_3t^2 + 2b_3t + c_3 = \lambda z \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Multiplying (1), (2), (3) by A_1, A_2, A_3 successively and adding, we have

$$t^2 . \Delta = \lambda (A_1x + A_2y + A_3z).$$

So $$2t . \Delta = \lambda (B_1x + B_2y + B_3z),$$

and $$\Delta = \lambda (C_1x + C_2y + C_3z).$$

$$\therefore (B_1x + B_2y + B_3z)^2 = 4(A_1x + A_2y + A_3z)(C_1x + C_2y + C_3z),$$

$$i.e.\ x^2(B_1^2 - 4A_1C_1) + \ldots + \ldots + 2yz[B_2B_3 - 2A_2C_3 - 2A_3C_2] + \ldots + \ldots = 0.$$

If the coordinates are Cartesian, we have z equal to unity.

142. *Equation to a tangent, and coordinates of the point of intersection of two tangents.*

The line joining the points t_1 and t_2 is (Art. 51)

$$\begin{vmatrix} x, & y, & z \\ a_1t_1^2 + 2b_1t_1 + c_1, & a_2t_1^2 + 2b_2t_1 + c_2, & a_3t_1^2 + 2b_3t_1 + c_3 \\ a_1t_2^2 + 2b_1t_2 + c_1, & a_2t_2^2 + 2b_2t_2 + c_2, & a_3t_2^2 + 2b_3t_2 + c_3 \end{vmatrix} = 0.$$

On subtracting the constituents of the second row from those of the third and dividing by $t_2 - t_1$, it becomes

$$\begin{vmatrix} x, & y, & z \\ a_1t_1^2 + 2b_1t_1 + c_1, & a_2t_1^2 + 2b_2t_1 + c_2, & a_3t_1^2 + 2b_3t_1 + c_3 \\ a_1(t_1 + t_2) + 2b_1, & a_2(t_1 + t_2) + 2b_2, & a_3(t_1 + t_2) + 2b_3 \end{vmatrix} = 0.$$

From the second row, multiplied by $t_1 + t_2$, subtract the third row multiplied by t_1^2, and we have

$$\begin{vmatrix} x, & y, & z \\ 2b_1 t_1 t_2 + c_1 (t_1 + t_2), & 2b_2 t_1 t_2 + c_2 (t_1 + t_2), & 2b_3 t_1 t_2 + c_3 (t_1 + t_2) \\ a_1 (t_1 + t_2) + 2b_1, & a_2 (t_1 + t_2) + 2b_2, & a_3 (t_1 + t_2) + 2b_3 \end{vmatrix} = 0 \qquad \text{......(1).}$$

On putting $t_2 = t_1$, the equation of the tangent at t_1 is

$$\begin{vmatrix} x, & y, & z \\ a_1 t_1 + b_1, & a_2 t_1 + b_2, & a_3 t_1 + b_3 \\ b_1 t_1 + c_1, & b_2 t_1 + c_2, & b_3 t_1 + c_3 \end{vmatrix} = 0.$$

So the tangent at t_2 is

$$\begin{vmatrix} x, & y, & z \\ a_1 t_2 + b_1, & a_2 t_2 + b_2, & a_3 t_2 + b_3 \\ b_1 t_2 + c_1, & b_2 t_2 + c_2, & b_3 t_2 + c_3 \end{vmatrix} = 0.$$

These clearly meet at the point

$$[a_1 t_1 t_2 + b_1 (t_1 + t_2) + c_1,\ a_2 t_1 t_2 + b_2 (t_1 + t_2) + c_2,\ a_3 t_1 t_2 + b_3 (t_1 + t_2) + c_3],$$

which is therefore the pole of the line joining the points t_1 and t_2.

In (1), the coefficient of x is, after simplification,

$$2 (a_2 b_3 - b_2 a_3) t_1 t_2 - (c_2 a_3 - c_3 a_2)(t_1 + t_2) + 2 (b_2 c_3 - b_3 c_2),$$

and so for those of y and z.

143. *To shew that the general equation of any conic can be reduced to a form which can be satisfied by equations of the form*

$$\frac{x}{a_1 t^2 + 2b_1 t + c_1} = \frac{y}{a_2 t^2 + 2b_2 t + c_2} = \frac{z}{a_3 t^2 + 2b_3 t + c_3}.$$

The general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is

$$(ax + hy + gz)^2 + (ab - h^2) y^2 + (ac - g^2) z^2 - 2 (gh - af) yz = 0,$$

i.e. $C (ax + hy + gz)^2 + (Cy - Fz)^2 + z^2 (BC - F^2) = 0,$

with the usual notation.

This equation is satisfied by

$$\sqrt{C}\,(ax+hy+gz)=\lambda\,(1-t^2),$$

$$Cy-Fz=\lambda\,.\,2t,$$

and

$$\sqrt{-(BC-F^2)}\,.\,z=\lambda\,(1+t^2).$$

On solving these, we have the values of x, y, z in the form given above.

The conic whose equation is given in the form

$$(l_1x+m_1y+n_1z)\,(l_2x+m_2y+n_2z)=(l_3x+m_3y+n_3z)^2$$

is satisfied by

$$\frac{l_1x+m_1y+n_1z}{1}=\frac{l_2x+m_2y+n_2z}{t^2}=\frac{l_3x+m_3y+n_3z}{t}$$

and hence, on solving, we have the ratios of x, y and z as above.

Numerical Example. The conic

$$x^2+2y^2+3z^2+2yz+4zx+6xy=0$$

is satisfied by

$$x=-1+14t-54t^2,\ y=2t,\ \text{and}\ z=1-10t+18t^2.$$

144. *Shew that the cross-ratio subtended at any point of the conic by the points t_1, t_2, t_3, t_4 is*

$$\frac{(t_2-t_1)\,(t_4-t_3)}{(t_3-t_2)\,(t_1-t_4)}.$$

Let O be the point where $t=0$ so that its coordinates are $(c_1,\ c_2,\ c_3)$.

Then OP_1 is

$$\begin{vmatrix} x, & y, & z \\ a_1t_1^2+2b_1t_1+c_1, & a_2t_1^2+2b_2t_1+c_2, & a_3t_1^2+2b_3t_1+c_3 \\ c_1, & c_2, & c_3 \end{vmatrix}=0,$$

i.e.

$$2\,[x\,(b_2c_3-b_3c_2)+y\,(b_3c_1-b_1c_3)+z\,(b_1c_2-b_2c_1)]$$
$$=t_1\,[x\,(c_2a_3-c_3a_2)+y\,(c_3a_1-c_1a_3)+z\,(c_1a_2-c_2a_1)],$$

and so for OP_2, OP_3, OP_4.

Hence, as in Art. 63, the cross-ratio is as stated.

145. *Cartesian Coordinates.*

The conic

$$\frac{x}{a_1t^2+2b_1t+c_1}=\frac{y}{a_2t^2+2b_2t+c_2}=\frac{1}{a_3t^2+2b_3t+c_3}$$

has points at infinity given by

$$a_3t^2+2b_3t+c_3=0 \quad \text{..................(1)},$$

and is thus an ellipse, parabola, or hyperbola according as $b_3^2 \lesseqgtr a_3c_3$.

From (1), the values of t giving the line at infinity are such that

$$\frac{t_1t_2}{c_3}=\frac{t_1+t_2}{-2b_3}=\frac{1}{a_3},$$

and hence, as in Art. 142, the centre, which is the pole of the line at infinity, is given by

$$\frac{x}{a_1c_3-2b_1b_3+c_1a_3}=\frac{y}{a_2c_3-2b_2b_3+c_2a_3}=\frac{1}{2(a_3c_3-b_3^2)}.$$

146. *Shew that the conic given by*

$$x=a_1t^2+2b_1t+c_1,\ y=a_2t^2+2b_2t+c_2$$

is a parabola, whose directrix is

$$a_1x+a_2y=a_1c_1-b_1^2+a_2c_2-b_2^2,$$

and whose latus-rectum is

$$4\frac{(a_1b_2-a_2b_1)^2}{(a_1^2+a_2^2)^{\frac{3}{2}}}.$$

As in Art. 141, the line

$$lx+my+n=0 \quad \text{......................(1)}$$

touches the conic if

$$l^2(b_1^2-a_1c_1)+m^2(b_2^2-a_2c_2)+lm(2b_1b_2-a_1c_2-a_2c_1)$$
$$-nla_1-mna_2=0 \text{...(2)}.$$

This equation is satisfied by $l=m=0$, *i.e.* by the line at infinity, and the conic is therefore a parabola.

If (x_1, y_1) be any point on (1), then $n=-lx_1-my_1$. On substituting this value in (2), the lines given by it are at right angles if the sum of the coefficients in the resulting equation is zero, *i.e.* if $a_1x_1+a_2y_1+b_1^2-a_1c_1+b_2^2-a_2c_2=0$, and the locus of (x_1, y_1), *i.e.* the directrix, is as stated.

If again (x_1, y_1) be the focus, this equation must reduce to $l^2 + m^2 = 0$, and hence we have

$$a_1 x_1 - a_2 y_1 = a_1 c_1 - b_1^2 - (a_2 c_2 - b_2^2),$$

and $$a_2 x_1 + a_1 y_1 = a_1 c_2 + a_2 c_1 - 2b_1 b_2,$$

i.e. $$(a_1^2 + a_2^2)(x_1 - c_1) = a_1 (b_2^2 - b_1^2) - 2a_2 b_1 b_2,$$

and $$(a_1^2 + a_2^2)(y_1 - c_2) = a_2 (b_1^2 - b_2^2) - 2a_1 b_1 b_2,$$

giving the coordinates of the focus.

The perpendicular from the focus upon the directrix

$$= \frac{a_1 (x_1 - c_1) + a_2 (y_1 - c_2) + b_1^2 + b_2^2}{(a_1^2 + a_2^2)^{\frac{1}{2}}} = \frac{2 (a_1 b_2 - a_2 b_1)^2}{(a_1^2 + a_2^2)^{\frac{3}{2}}},$$

and the latus-rectum is twice this perpendicular.

EXAMPLES.

1. If $$x : y : z :: t^2 + 8t + 2 : 4t^2 + 5t + 1 : 4t^2 - t + 1,$$
find the tangential equation of the conic, and prove that it is a parabola, the coordinates being areal.

2. Shew that $\dfrac{x}{5 - 6t - 3t^2} = \dfrac{y}{5 + 8t - t^2} = \dfrac{1}{1 + t^2}$ represents a circle of radius 5.

3. Find the focus and directrix of the parabola
$$x = t^2 - 2t + 2, \quad y = t^2 + 1.$$

4. Find the foci and the director circle of the conic of Art. 145.

5. In the parabola $x = a_1 t^2 + 2b_1 t + c_1$, $y = a_2 t^2 + 2b_2 t + c_2$, prove that the values of "t" at the ends of the latus rectum are
$$\frac{\pm (a_1 b_2 - a_2 b_1) - (a_1 b_1 + a_2 b_2)}{a_1^2 + a_2^2}.$$

6. If t_1 and t_2 are the parameters of the ends of any chord parallel to $y = mx$ of the parabola $x = a_1 t^2 + 2b_1 t + c_1$, $y = a_2 t^2 + 2b_2 t + c_2$, shew that
$$t_1 + t_2 = -2 \frac{b_2 - mb_1}{a_2 - ma_1}.$$

7. Shew that the parametric form in areal coordinates of the conic which passes through the middle points D, E, F of the sides of the triangle of reference, through the vertex C, and which touches AB at F, is
$$x : y : z :: 2 - i + 4t + (2 + i) t^2 : i (1 - t^2) : -4i + 8t + 4it^2,$$
where $i = \sqrt{-1}$.

CHAPTER V

TANGENTIAL COORDINATES

147. If in the general equation to a straight line

$$l\alpha + m\beta + n\gamma = 0$$

we give to l, m, n particular values, we get a definite and determinate straight line. Hence l, m, n are called the coordinates of the line because they determine its position, and they are denoted by the symbol (l, m, n).

If there is a linear relation between these line-coordinates, such as $l\alpha_1 + m\beta_1 + n\gamma_1 = 0$, where α_1, β_1, γ_1 are constants, then the straight line passes through the point $(\alpha_1, \beta_1, \gamma_1)$. Such a linear relation between the line coordinates therefore determines a point and is the equation to a point.

[Thus the tangential equation of the circum-centre is

$$l \cos A + m \cos B + n \cos C = 0,$$

this being the condition that the straight line $l\alpha + m\beta + n\gamma = 0$ should pass through the circum-centre whose coordinates are

$$(\cos A, \cos B, \cos C).$$

So the tangential equations of the ortho-centre, in-centre and centroid are respectively

$$l \sec A + m \sec B + n \sec C = 0, \quad l + m + n = 0,$$

and $$l \operatorname{cosec} A + m \operatorname{cosec} B + n \operatorname{cosec} C = 0.$$

If there is a relation of the second degree between l, m, and n, we have shewn in Art. 72 that the line envelopes a conic; in other words a tangential equation of the second degree represents a conic.

If there is an algebraic homogeneous relation of any kind between l, m, and n, the straight line will envelope some curve, and this algebraic relation is called the tangential equation of that curve.

148. The foregoing will hold similarly if we use Areal Coordinates. But, in order to shew what coordinates are meant, the letters p, q, r will be used to denote the coordinates of a line in Areal Coordinates, and hence

$$px + qy + rz = 0$$

will be the standard equation of a line in Areal Coordinates.

Most of the Articles in the present chapter will be true for Areal as well as Trilinear Coordinates. Where there is any difference it will be mentioned.

If we use Cartesian Coordinates, we shall have the coordinate z equal to unity, and then x and y are the Cartesian Coordinates.

149. We can easily convert tangential equations in trilinear coordinates into the similar equations in areal coordinates. For suppose that the trilinear equation

$$l\alpha + m\beta + n\gamma = 0$$

represents the same straight line as the areal equation

$$px + qy + rz = 0,$$

so that $l\alpha$, $m\beta$, $n\gamma$ are proportional to px, qy, rz. Now α, β, γ are, as in Art. 67, proportional to $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$. Hence $\frac{l}{a}, \frac{m}{b}, \frac{n}{c}$ are proportional to p, q, r and hence l, m, n to pa, qb, rc. It follows that a tangential equation in trilinear coordinates can be changed into the corresponding one in areal coordinates by substituting pa, qb, and rc for l, m and n, where a, b and c are the sides of the triangle of reference, and conversely.

150. The equation of any straight line through the intersection of $l_1\alpha + m_1\beta + n_1\gamma = 0$ and $l_2\alpha + m_2\beta + n_2\gamma = 0$ is

$$(l_1 + \lambda l_2)\,\alpha + (m_1 + \lambda m_2)\,\beta + (n_1 + \lambda n_2)\,\gamma = 0,$$

so that the tangential coordinates of any line through the intersections of the lines (l_1, m_1, n_1) and (l_2, m_2, n_2) are $(l_1 + \lambda l_2,\ m_1 + \lambda m_2,\ n_1 + \lambda n_2)$.

The tangential coordinates of the line joining two fixed points $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ are, as in Art. 51,

$$(\beta_1\gamma_2 - \beta_2\gamma_1,\ \gamma_1\alpha_2 - \gamma_2\alpha_1,\ \alpha_1\beta_2 - \alpha_2\beta_1).$$

151. Just as in Part I, Art. 116, it may be shewn that the tangential equation

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

breaks up into linear factors if

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0.$$

In this case the conic becomes a pair of points. For let the two linear factors be $l\alpha_1 + m\beta_1 + n\gamma_1$ and $l\alpha_2 + m\beta_2 + n\gamma_2$. Then $\Sigma = 0$ is satisfied when either $l\alpha_1 + m\beta_1 + n\gamma_1 = 0$, or when $l\alpha_2 + m\beta_2 + n\gamma_2 = 0$, *i.e.* when the straight line

$$l\alpha + m\beta + n\gamma = 0$$

passes through either of the points $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$. Hence these two points only are represented.

152. The tangential equation of a circle, whose radius is r and whose centre is at the point $(\alpha', \beta', \gamma')$, is easily obtained by expressing the condition that the perpendicular from the centre upon any tangent is equal to r, and is thus, by Art. 59,

$$\frac{l\alpha' + m\beta' + n\gamma'}{\sqrt{l^2 + \ldots + \ldots - 2mn\cos A - \ldots - \ldots}} = r = \frac{a\alpha' + b\beta' + c\gamma'}{2\Delta} \times r,$$

i.e.

$$r^2(a\alpha' + b\beta' + c\gamma')^2(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C)$$
$$= 4\Delta^2(l\alpha' + m\beta' + n\gamma')^2.$$

So in areal coordinates it is

$$r^2(x' + y' + z')^2(a^2p^2 + b^2q^2 + c^2r^2 - 2qr\,bc\cos A - \ldots - \ldots)$$
$$= 4\Delta^2(px' + qy' + rz')^2,$$

where in each case a, b, and c are the sides of the triangle of reference.

153. *To find the point of contact of any tangent to a conic given by its tangential equation.*

Let the point equation to a conic be

$$S \equiv a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \ldots(1),$$

so that its tangential equation is

$$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \ldots(2),$$

where $A = bc - f^2$, ..., ..., $F = gh - af$, ..., ... as in Art. 72.

Let the point of contact of the tangent

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

be the point $(\alpha_1, \beta_1, \gamma_1)$. Then (3) must be the same as the tangent to (1) at $(\alpha_1, \beta_1, \gamma_1)$, *i.e.* it is the same as

$$\alpha(a\alpha_1 + h\beta_1 + g\gamma_1) + \beta(h\alpha_1 + b\beta_1 + f\gamma_1) + \gamma(g\alpha_1 + f\beta_1 + c\gamma_1) = 0.$$

$$\therefore\ a\alpha_1 + h\beta_1 + g\gamma_1 = \lambda l_1,$$

$$h\alpha_1 + b\beta_1 + f\gamma_1 = \lambda m_1,$$

and

$$g\alpha_1 + f\beta_1 + c\gamma_1 = \lambda n_1.$$

Multiplying these equations by A, H, G and adding, we have

$$\Delta\alpha_1 = \lambda[Al_1 + Hm_1 + Gn_1],$$

and so for the others.

$$\therefore\ \frac{\alpha_1}{Al_1 + Hm_1 + Gn_1} = \frac{\beta_1}{Hl_1 + Bm_1 + Fn_1} = \frac{\gamma_1}{Gl_1 + Fm_1 + Cn_1},$$

giving the required point of contact.

But the tangential equation to the point $(\alpha_1, \beta_1, \gamma_1)$ is

$$l\alpha_1 + m\beta_1 + n\gamma_1 = 0.$$

Hence the tangential equation of the point of contact of the line (l_1, m_1, n_1) is

$$l[Al_1 + Hm_1 + Gn_1] + m[Hl_1 + Bm_1 + Fn_1] + n[Gl_1 + Fm_1 + Cn_1] = 0,$$

i.e. $$l\frac{d\Sigma}{dl_1} + m\frac{d\Sigma}{dm_1} + n\frac{d\Sigma}{dn_1} = 0, \text{ or } l_1\frac{d\Sigma}{dl} + m_1\frac{d\Sigma}{dm} + n_1\frac{d\Sigma}{dn} = 0.$$

Either of these is of the same form as the tangent at a given point in point coordinates (Art. 71).

154. *To find the pole of any line (l_1, m_1, n_1) with respect to the conic whose tangential equation is*

$$\Sigma = Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

Let the line $l_1\alpha + m_1\beta + n_1\gamma = 0$ meet the conic in two points the tangents at which are

$$l_2\alpha + m_2\beta + n_2\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and

$$l_3\alpha + m_3\beta + n_3\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

By the last article, the point of contact of (1) is

$$(Al_2+Hm_2+Gn_2,\ \ Hl_2+Bm_2+Fn_2,\ \ Gl_2+Fm_2+Cn_2).$$

Since this point lies on the line $l_1\alpha+m_1\beta+n_1\gamma=0$,

$$\therefore\ l_1(Al_2+Hm_2+Gn_2)+m_1(Hl_2+Bm_2+Fn_2)$$
$$+n_1(Gl_2+Fm_2+Cn_2)=0,$$

or

$$l_2(Al_1+Hm_1+Gn_1)+m_2(Hl_1+Bm_1+Fn_1)$$
$$+n_2(Gl_1+Fm_1+Cn_1)=0.$$

Similarly

$$l_3(Al_1+Hm_1+Gn_1)+m_3(Hl_1+Bm_1+Fn_1)$$
$$+n_3(Gl_1+Fm_1+Cn_1)=0.$$

But these are the conditions that the two tangents pass through the point

$$(Al_1+Hm_1+Gn_1,\ \ Hl_1+Bm_1+Fn_1,\ \ Gl_1+Fm_1+Cn_1),$$

which is therefore the required pole of the line $(l_1,\ m_1,\ n_1)$. This is the result of Art. 76.

These coordinates may be written in the form

$$\left(\frac{d\Sigma}{dl_1},\ \ \frac{d\Sigma}{dm_1},\ \ \frac{d\Sigma}{dn_1}\right).$$

The tangential equation of this pole is thus

$$l\frac{d\Sigma}{dl_1}+m\frac{d\Sigma}{dm_1}+n\frac{d\Sigma}{dn_1}=0\ \text{ or }\ l_1\frac{d\Sigma}{dl}+m_1\frac{d\Sigma}{dm}+n_1\frac{d\Sigma}{dn}=0.$$

The forms of these equations are similar to those of Art. 75.

155. Two lines $(l_1,\ m_1,\ n_1)$ and $(l_2,\ m_2,\ n_2)$ are conjugate if the pole of each lies on the other, *i.e.* if

$$l_1\frac{d\Sigma}{dl_2}+m_1\frac{d\Sigma}{dm_2}+n_1\frac{d\Sigma}{dn_2}=0,$$

or

$$l_2\frac{d\Sigma}{dl_1}+m_2\frac{d\Sigma}{dm_1}+n_2\frac{d\Sigma}{dn_1}=0.$$

This was also found in Art. 76.

The line $(l,\ m,\ n)$ is a tangent if its pole lies on itself, *i.e.* if

$$l\frac{d\Sigma}{dl}+m\frac{d\Sigma}{dm}+n\frac{d\Sigma}{dn}=0,$$

i.e. if $\Sigma=0$.

156. *Centre and asymptotes of a conic given by the general tangential equation in trilinear coordinates.*

The centre is the pole of the line at infinity

$$a\alpha + b\beta + c\gamma = 0.$$

Hence, as in the last article, the centre is the point

$$(Aa + Hb + Gc,\ Ha + Bb + Fc,\ Ga + Fb + Cc).$$

Also the asymptotes are the pair of tangents that can be drawn from the centre to the curve.

Hence an asymptote is the line (l, m, n) given by

$$l(Aa + Hb + Gc) + m(Ha + Bb + Fc) + n(Ga + Fb + Cc) = 0 \quad \text{......(1)},$$

and

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \quad \text{...(2)},$$

i.e. if $\phi(l, m, n) = 0$ is the tangential equation, the centre is given by

$$a\frac{d\phi}{dl} + b\frac{d\phi}{dm} + c\frac{d\phi}{dn} = 0,$$

and the asymptotes by this equation and $\phi(l, m, n) = 0$.

On solving these two equations, we have the asymptotes.

The conic is an ellipse, parabola, or hyperbola according as the roots are real, coincident, or imaginary.

The condition that the conic may be a parabola is more easily obtained from the fact that the line at infinity is a tangent, and hence (2) is satisfied by $l = a$, $m = b$ and $n = c$. The condition is thus

$$Aa^2 + Bb^2 + Cc^2 + 2Fbc + 2Gca + 2Hab = 0.$$

157. In **Areal** Coordinates the centre is the pole of the line $x + y + z = 0$, and is given by

$$\frac{d\phi}{dp} + \frac{d\phi}{dq} + \frac{d\phi}{dr} = 0,$$

so that it is

$$(A + H + G,\ H + B + F,\ G + F + C).$$

The asymptotes are given by

$$p(A + H + G) + q(H + B + F) + r(G + F + C) = 0,$$

and

$$Ap^2 + Bq^2 + Cr^2 + 2Fqr + 2Grp + 2Hpq = 0,$$

i.e. by

$$\frac{d\phi}{dp} + \frac{d\phi}{dq} + \frac{d\phi}{dr} = 0 \text{ and } \phi(p, q, r) = 0.$$

The conic is a parabola if

$$A + B + C + 2F + 2G + 2H = 0.$$

158. *To find the equation to the director circle of the conic whose tangential equation is*

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

If the line (l, m, n) passes through the point $(\alpha_1, \beta_1, \gamma_1)$, then $l\alpha_1 + m\beta_1 + n\gamma_1 = 0$.

If we eliminate n between this and the tangential equation, we obtain

$$l^2[A\gamma_1^2 - 2G\gamma_1\alpha_1 + C\alpha_1^2] + 2lm[H\gamma_1^2 - G\beta_1\gamma_1 - F\gamma_1\alpha_1 + C\alpha_1\beta_1] + m^2[B\gamma_1^2 - 2F\beta_1\gamma_1 + C\beta_1^2] = 0.$$

This equation gives the two tangents, (l_1, m_1, n_1) and (l_2, m_2, n_2), which pass through $(\alpha_1, \beta_1, \gamma_1)$. Hence

$$\frac{l_1 l_2}{B\gamma_1^2 - 2F\beta_1\gamma_1 + C\beta_1^2} = \frac{m_1 m_2}{C\alpha_1^2 - 2G\gamma_1\alpha_1 + A\gamma_1^2} = \frac{l_1 m_2 + l_2 m_1}{-2H\gamma_1^2 + 2G\beta_1\gamma_1 + 2F\gamma_1\alpha_1 - 2C\alpha_1\beta_1}.$$

We should, by eliminating l and m instead of n, obtain similar relations for $n_1 n_2$, $m_1 n_2 + m_2 n_1$, and $n_1 l_2 + n_2 l_1$.

Now if $(\alpha_1, \beta_1, \gamma_1)$ is on the director circle the two tangents are at right angles. Hence, on substitution in the result of Art. 54, we have

$$(B\gamma_1^2 - 2F\beta_1\gamma_1 + C\beta_1^2) + \ldots + \ldots + 2[F\alpha_1^2 + A\beta_1\gamma_1 - H\gamma_1\alpha_1 - G\alpha_1\beta_1]\cos A + \ldots + \ldots = 0.$$

Hence the equation to the director circle is, as in Art. 79,

$$\alpha^2(B + C + 2F\cos A) + \ldots + \ldots + 2\beta\gamma[A\cos A - F - G\cos C - H\cos B] + \ldots + \ldots.$$

Ex. 1. Shew that the equation to the director circle of a conic, whose tangential equation in oblique Cartesian coordinates is

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,$$

is $C(x^2+2xy\cos\omega+y^2)$

$$-2(G+F\cos\omega)x-2(F+G\cos\omega)y+A+B+2H\cos\omega=0.$$

Ex. 2. With rectangular Cartesian coordinates shew that the tangential equation

$$Al^2+Bm^2+2Fmn+2Gnl+2Hlm=0$$

is that of a parabola, whose directrix is given by $2Gx+2Fy=A+B$, and whose focus (x_1, y_1) is given by $2(x_1+iy_1)(G+Fi)=A-B+2Hi$, where $i\equiv\sqrt{-1}$. [A tangent through the focus is $y-y_1=i(x-x_1)$, so that the values $l=i, m=-1, n=y_1-ix$ satisfy the tangential equation.]

159. *To obtain the foci of a conic whose tangential equation in rectangular Cartesian Coordinates is*

$$\Sigma\equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

Let (x_1, y_1), (x_2, y_2) be the two real foci, or the two imaginary foci, of the curve. The product of the perpendiculars from them upon any tangent is constant.

Hence

$$\frac{(lx_1+my_1+n)(lx_2+my_2+n)}{l^2+m^2}=\text{const.}=\lambda.$$

Comparing this with the given tangential equation, we have

$$\frac{x_1x_2-\lambda}{A}=\frac{y_1y_2-\lambda}{B}=\frac{1}{C}=\frac{y_1+y_2}{2F}=\frac{x_1+x_2}{2G}=\frac{x_1y_2+x_2y_1}{2H},$$

$$\therefore\ C(x_1x_2-y_1y_2)=A-B;\ \ C(x_1+x_2)=2G;$$

$$C(y_1+y_2)=2F;\ \text{and}\ \ C(x_1y_2+x_2y_1)=2H.$$

Hence, substituting for x_2 and y_2, we have x_1, y_1 given by

$$C(x_1^2-y_1^2)-2Gx_1+2Fy_1+A-B=0,$$

and

$$Cx_1y_1-Fx_1-Gy_1+H=0,$$

two equations to give the foci. They represent two rectangular hyperbolas which intersect in the two real, and the two imaginary, foci.

Again, since

$$(lx_1+my_1+n)(lx_2+my_2+n)=\Sigma+\lambda(l^2+m^2),$$

the right-hand member must break into factors, the condition for which is

$$C(A+\lambda)(B+\lambda)+2FGH \\ -(A+\lambda)F^2-(B+\lambda)G^2-(C+\lambda)H^2=0,$$

or

$$C\lambda^2+\lambda[(A+B)C-F^2-G^2] \\ +(ABC+2FGH-AF^2-BG^2-CH^2)=0.$$

Now, if the foci are real, the value of λ is the square of the semi-minor axis; whilst, if the foci are imaginary, it can be shewn (from Part I, Art. 394) that λ is equal to the square of the semi-major axis. Hence the squares of the semi-axes are given by the equation

$$C\lambda^2+\lambda[(A+B)C-F^2-G^2]+\Delta=0,$$

where Δ is the discriminant of the tangential equation.

160. Ex. *Conics are drawn touching the sides of the triangle of reference and also the straight line* $(1, -1, 1)$; *shew that the tangential equation of the envelope of the asymptotes is* $q(p+q+r)-pr=0$, *and interpret the result.*

The tangential equation of an inscribed conic is

$$Lqr+Mrp+Npq=0.$$

This is satisfied by $(1, -1, 1)$ if $M=L+N$. Hence, on putting $N=\mu L$, the equation is

$$qr+(\mu+1)rp+\mu pq=0 \quad \text{.........................} (1).$$

The centre is (Art. 157)

$$p(2\mu+1)+q(\mu+1)+r(\mu+2)=0 \quad \text{...............} (2).$$

The asymptotes are given by (1) and (2). Eliminating μ, we have

$$(p-r)[q(p+q+r)-pr]=0.$$

This is therefore the tangential equation always satisfied by the asymptotes. The factor $p-r=0$ gives $p=q=r=0$.

Hence the equation required is

$$q(p+q+r)-pr=0 \quad \text{........................} (3).$$

This is satisfied by $p=q=0$, and by $q=r=0$. Hence the envelope of (3) is a conic which touches the sides $z=0$ and $x=0$.

By Art. 73, the point-equation is easily seen to be

$$x^2+y^2+z^2+2yz-6zx+2xy=0.$$

EXAMPLES.

1. Shew by a diagram the position of the straight line whose areal coordinates are proportional to $(-1, 2, 3)$, and of the point whose equation is $-p+2q+3r=0$.

2. In areal coordinates, shew that the tangential equation to a conic touching the sides of the triangle of reference, and having its centre at (x_0, y_0, z_0), is

$$(y_0+z_0-x_0)\,qr+(z_0+x_0-y_0)\,rp+(x_0+y_0-z_0)\,pq=0.$$

If the centre of the conic is the nine-point centre, shew that the equation is

$$qr\sin 2A+rp\sin 2B+pq\sin 2C=0.$$

Shew that $qr\cot\frac{A}{2}+rp\cot\frac{B}{2}+pq\cot\frac{C}{2}=0$ is the tangential equation of the circle inscribed in the triangle of reference.

3. In areal coordinates, what curve is represented by the tangential equation

$$(ap+bq-cr)(ap-bq+cr)+(ap+bq-cr)(-ap+bq+cr)$$
$$+(-ap+bq+cr)(ap-bq+cr)=0?$$

4. In areal coordinates, shew what is represented by the equations

(1) $p\tan A+q\tan B+r\tan C=0$;

(2) $a^2p^2=4bc\,qr$;

(3) $2p^2=(p+q)\,r$;

(4) $a\sqrt{p}+b\sqrt{q}+c\sqrt{r}=0$;

(5) $a\sqrt{q+r}+b\sqrt{r+p}+c\sqrt{p+q}=0$;

where a, b, c are the sides of the fundamental triangle.

5. Find the centre and asymptotes of the conic whose tangential equation is $p^2+pq-2q^2-r^2=0$.

6. Find the coordinates of the points of contact of the sides of the triangle of reference with the conic whose tangential equation is

$$Lqr+Mrp+Npq=0.$$

Find also the centre of the conic.

7. Shew that the straight lines joining the vertices of the fundamental triangle to the points in which the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

cuts the opposite sides touch a conic whose tangential equation is

$$bcp^2+caq^2+abr^2-2af\,qr-2bg\,rp-2ch\,pq=0.$$

8. Shew that the three perpendiculars from the vertices upon the opposite sides of the triangle of reference, and the three lines to the middle points of these sides from the vertices, all touch the conic whose tangential equation is

$$l^2\sin 2B\sin 2C+\ldots+\ldots+2mn\sin A\sin 2A+\ldots+\ldots=0.$$

9. Find the tangential equations of the inscribed and nine-point circles of the triangle of reference, and shew that one of their centres of similitude lies on them.

10. Shew that all conics of the system $p^2-q^2=kr^2$, where k is a variable parameter, have one common asymptote, and that their other asymptotes all pass through a common point.

11. If p and q are the lengths of the perpendiculars on a variable line from two fixed points, shew that

$$ap+bq+c=0 \quad \text{and} \quad ap^2+2hpq+bq^2=c^2$$

represent respectively a circle and a conic.

Interpret the equation $p'q+q'p=2k^2$, where the line (p', q') is any tangent to the ellipse $pq=k^2$.

12. Shew that the lines joining the vertices of the triangle of reference to the poles of the opposite sides with respect to the conic, whose tangential equation is

$$Ap^2+Bq^2+Cr^2+2Fqr+2Grp+2Hpq=0,$$

meet at the point $\qquad Fx=Gy=Hz.$

If this conic touch four fixed lines, shew that the locus of this point is a conic circumscribing the triangle of reference.

13. If the equation to a straight line in rectangular Cartesian coordinates be $lx+my+n=0$, shew that the equation $4c^2lm=n^2$ represents a rectangular hyperbola, and that the equation to its evolute is

$$c^2(l^2-m^2)^2+lmn^2=0.$$

14. Shew that the equation to an asymptote of the conic $yz=kx^2$ is

$$4kx+y+z=\pm(y-z)\sqrt{1-4k},$$

and shew that for all conics of the system its envelope is the parabola

$$(y+z+2x)^2=4yz.$$

15. Two tangents are drawn from a fixed point O to a conic; the first is met by a variable tangent in P, and the second by a parallel tangent in Q. Shew that the rectangle $OP.OQ$ is constant. [This may be solved by either tangential or homographic methods.]

16. Shew that tangents to the conic

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

at the points where it is met by the line $px+qy+rz=0$, are given by the equation

$$S.\Sigma=\Delta(px+qy+rz)^2,$$

where $\Sigma=0$ is the tangential equation of $S=0$, and Δ is the discriminant of S.

Meaning of $\Sigma+\lambda\Sigma'=0$.

161. If $\Sigma=0$ and $\Sigma'=0$ are the tangential equations to two conics (*i.e.* the conditions that the general straight line $l\alpha+m\beta+n\gamma=0$ touches them), then $\Sigma+\lambda\Sigma'=0$ is the tangential equation to a conic which, for all values of λ, touches the four common tangents of Σ and Σ'. For it is satisfied by any straight line whose line-coordinates satisfy both $\Sigma=0$ and $\Sigma'=0$, *i.e.* by any common tangent to both conics.

[The student should notice the analogy with the point equation $S+\lambda S'=0$ (Part I, Art. 380). The conic $S+\lambda S'=0$ passes through all the points whose coordinates satisfy both $S=0$ and $S'=0$. The conic $\Sigma+\lambda\Sigma'=0$ touches all the lines whose coordinates satisfy both $\Sigma=0$ and $\Sigma'=0$.]

Particular Cases.

$\Sigma+\lambda PQ=0$, where P and Q are linear, *i.e.* they are the tangential equations of points.

As in the general case this case gives a conic whose equation is satisfied by $\Sigma=0$ and $P=0$, or by $\Sigma=0$ and $Q=0$, *i.e.* it touches all the lines which touch $\Sigma=0$ and pass through the two points given by $P=0$ and $Q=0$, *i.e.* it touches the four tangents which can be drawn from the points $P=0$ and $Q=0$ to the conic $\Sigma=0$.

$\Sigma+\lambda P^2=0$. If the point $Q=0$ moves up to, and ultimately coincides with, the point $P=0$, the point of contact of the conic with the tangent from the point $Q=0$ will move up to, and ultimately coincide with, the point of contact of the tangent from $P=0$, *i.e.* the conic $\Sigma+\lambda P^2=0$ will touch the conic $\Sigma=0$ at the points where the tangents from the point $P=0$ meet it.

$PQ+\lambda RS=0$, where $P=0$, $Q=0$, $R=0$, $S=0$ are the tangential equations of points A, B, C, D.

The equation is satisfied by any line which satisfies both $P=0$ and $R=0$, *i.e.* by the line AC; so by any line which satisfies $P=0$ and $S=0$, *i.e.* by the line AD; similarly by a line which satisfies $Q=0$, $R=0$, or $Q=0$, $S=0$, *i.e.* by the lines BC or BD. Hence the equation gives any conic which touches the four lines AC, AD, BC and BD.

If P and Q coincide, then $P^2 + \lambda RS = 0$ is the tangential equation of a conic which touches the lines AC and AD at C and D, where A, C, D are the points whose tangential equations are $P = 0$, $R = 0$, and $S = 0$.

162. Ex. *If p_1, p_2, p_3, p_4 are the perpendiculars from the angular points of a quadrilateral upon any tangent to a conic inscribed in the quadrilateral, then $p_1p_3 = kp_2p_4$.*

Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ and $(\alpha_4, \beta_4, \gamma_4)$ be the coordinates of the vertices A, B, C, D of the quadrilateral.

Then $l\alpha_1 + m\beta_1 + n\gamma_1 = 0$ etc. being the tangential equations of the points A, B, C, D, the tangential equation of any conic touching AB, AD, CB, CD, *i.e.* of any inscribed conic, is, by the last article,

$$(l\alpha_1 + m\beta_1 + n\gamma_1)(l\alpha_3 + m\beta_3 + n\gamma_3) = k(l\alpha_2 + m\beta_2 + n\gamma_2)(l\alpha_4 + m\beta_4 + n\gamma_4).$$

But $l\alpha_1 + m\beta_1 + n\gamma_1$ is proportional to the perpendicular p_1, and similarly for the others. Hence we have

$$p_1p_3 = kp_2p_4.$$

163. Tangential equation of the circular points at infinity.

The circular points at infinity are given by the intersection of the circle

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0 \quad \text{.................(1)}$$

with the line at infinity

$$a\alpha + b\beta + c\gamma = 0 \quad \text{.................(2).}$$

The straight line

$$l\alpha + m\beta + n\gamma = 0 \quad \text{.................(3)}$$

passes through one or other of these points if these three loci have a common point.

(2) and (3) meet where

$$\frac{\alpha}{mc - nb} = \frac{\beta}{na - lc} = \frac{\gamma}{lb - ma},$$

and this point lies on (1) if

$$a(na-lc)(lb-ma) + b(lb-ma)(mc-nb) + c(mc-nb)(na-lc) = 0,$$

i.e. if

$$abc(l^2 + m^2 + n^2) - mn\,a(b^2 + c^2 - a^2) - nl\,b(c^2 + a^2 - b^2) - lm\,c(a^2 + b^2 - c^2) = 0,$$

i.e. if

$$l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C = 0 \ldots(4).$$

This equation, being the condition that (3) passes through one of the circular points at infinity, is their tangential equation.

164. If the coordinates be Cartesian and rectangular, the corresponding condition that $\lambda x + \mu y + \nu = 0$ should pass through the circular points at infinity is $\lambda^2 + \mu^2 = 0$. For the condition (4) implies (Art. 59) that for every line through these circular points the perpendicular on it from any point is infinite, and $\lambda^2 + \mu^2 = 0$ means this in Cartesian rectangular coordinates.

Or, again, the straight lines through any point to the circular points at infinity are $y \pm x\sqrt{-1} + c = 0$. Hence for these straight lines, $\dfrac{\lambda}{\sqrt{-1}} = \dfrac{\mu}{1}$, and so $\lambda^2 + \mu^2 = 0$.

Similarly, for oblique Cartesian coordinates, the corresponding equation is $\lambda^2 + 2\lambda\mu\cos\omega + \mu^2 = 0$, where ω is the angle between the axes.

If the coordinates are **Areal**, the circular points at infinity are given by the intersection of the circle

$$a^2yz + b^2zx + c^2xy = 0, \text{ and } x + y + z = 0,$$

and hence their tangential equation is

$$a^2p^2 + b^2q^2 + c^2r^2 - 2bcqr\cos A - 2carp\cos B - 2abpq\cos C = 0.$$

Tangential equation to the foci of a conic and to confocal conics.

165. We have seen in Part I, Art. 392, that the (imaginary) tangents from the focus of a conic to the conic satisfy the conditions for being a circle, and hence they pass through the circular points at infinity through which all circles pass.

If then from the circular points at infinity we draw tangents (imaginary) to the conic they form a quadrilateral two of whose angular points are the real foci, and the other two of whose angular points are the imaginary foci, of the conic. Also all conics which have the same tangents from the circular points at infinity have the same foci.

If $\Sigma = 0$ is the tangential equation of any conic, and

$$\Sigma' \equiv l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C = 0$$

is the tangential equation of the circular points at infinity, then

$$\Sigma + \lambda\Sigma' = 0$$

is satisfied by all lines which satisfy both Σ and $\Sigma'=0$, *i.e.* it is the tangential equation of all conics which touch the four tangents that can be drawn to $\Sigma=0$ from the circular points at infinity.

But the other intersections of these four tangents are the foci of the curve.

Hence $\Sigma+\lambda\Sigma'=0$ has the same foci as $\Sigma=0$, and thus the tangential equation of all conics confocal with $\Sigma=0$ is

$$\Sigma+\lambda\left[l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C\right]=0 \quad \ldots\ldots(1).$$

If this equation (1) breaks up into linear factors, and this by Art. 151, it will be found to do for two values of λ, it represents two pairs of points, and these points are the foci themselves.

166. Similarly, in **Areal** coordinates, the equation of all conics confocal with $\Sigma=0$ is

$$\Sigma+\lambda\left(a^2p^2+b^2q^2+c^2r^2-2bcqr\cos A-2carp\cos B-2abpq\cos C\right)=0.$$

167. *Tangential equation to the foci in Cartesian coordinates.*

If $\Sigma=0$ be the tangential equation to any conic, that to any confocal is

$$\Sigma+\lambda\left(l^2+m^2\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

i.e.

$$(A+\lambda)l^2+(B+\lambda)m^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

This breaks into factors, which are the tangential equations to the foci, if

$$(A+\lambda)(B+\lambda)C+2FGH-(A+\lambda)F^2-(B+\lambda)G^2-CH^2=0,$$

i.e. if

$$\left(ABC+2FGH-AF^2-BG^2-CH^2\right)+\lambda\left(AC+BC-F^2-G^2\right)+\lambda^2C=0,$$

i.e. if $$\Delta^2+\lambda(a+b)\Delta+\lambda^2C=0,$$

where Δ is the discriminant of the point equation.

Eliminating λ between this and (1), we have

$$\left(l^2+m^2\right)^2\Delta^2-\Sigma\Delta(a+b)\left(l^2+m^2\right)+C\Sigma^2=0,$$

as the required tangential equation of the foci.

It will be noted that in the above work it has been assumed that

$$(ABC+2FGH-AF^2-BG^2-CH^2)=(abc+2fgh-af^2-bg^2-ch^2)^2=\Delta^2.$$

This can be proved by the use of the relations of Art. 73. For the left hand

$$=A(BC-F^2)+H(FG-CH)+G(HF-BG)$$
$$=A\,.\,a\Delta+H\,.\,h\Delta+G\,.\,g\Delta=\Delta(Aa+Hh+Gg)=\Delta^2.$$

168. *General tangential equation to all conics having the given points* S_1, $(\alpha_1, \beta_1, \gamma_1)$, *and* S_2, $(\alpha_2, \beta_2, \gamma_2)$, *as foci.*

If $l\alpha+m\beta+n\gamma=0$ be any tangent, the product of the perpendiculars from the foci upon it is constant. Hence

$$\frac{(l\alpha_1+m\beta_1+n\gamma_1)(l\alpha_2+m\beta_2+n\gamma_2)}{l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C}=\text{const.}=\lambda.$$

The tangential equation required is thus

$$(l\alpha_1+m\beta_1+n\gamma_1)(l\alpha_2+m\beta_2+n\gamma_2)$$
$$=\lambda\left[l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C\right]\ldots(1).$$

If we want the parabola with S_1 as focus, then S_2 is the point of contact of the conic with the line at infinity, and hence we have in addition

$$a\alpha_2+b\beta_2+c\gamma_2=0.$$

Similarly, in **Areal** Coordinates, the tangential equation of all conics having (x_1, y_1, z_1) and (x_2, y_2, z_2) as foci is

$$(px_1+qy_1+rz_1)(px_2+qy_2+rz_2)$$
$$=\lambda(a^2p^2+b^2q^2+c^2r^2-2bcqr\cos A-2carp\cos B-2abpq\cos C),$$

and, if the conic is to be a parabola, we have, in addition,

$$x_2+y_2+z_2=0.$$

169. *If* $\phi\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ *be the general Cartesian equation to any conic, shew that the equation of all conics confocal with it is*

$$\phi+k\psi+k^2\Delta=0,$$

where $\psi\equiv C(x^2+y^2)-2Gx-2Fy+A+B$,

Δ *is the discriminant of* ϕ, *and* k *is a variable parameter.*

By Art 167, the tangential equation of any conic confocal with $\phi=0$ is

$$(A+\lambda)l^2+(B+\lambda)m^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

Hence, by Art. 73, the corresponding point-equation is

$$\begin{aligned}x^2[BC-F^2+\lambda C]+y^2[CA-G^2+\lambda C]+[(A+\lambda)(B+\lambda)-H^2]\\ +2y[GH-AF-\lambda F]+2x[HF-BG-\lambda G]\\ +2xy[FG-CH]=0\ldots(1).\end{aligned}$$

But, as in the same article, we have

$$\begin{aligned}&BC-F^2=a\Delta;\quad CA-G^2=b\Delta;\quad AB-H^2=c\Delta;\\ &GH-AF=f\Delta;\quad HF-BG=g\Delta;\quad \text{and } FG-CH=h\Delta,\end{aligned}$$

where Δ is the discriminant of the point-equation.

Hence (1) becomes

$$\Delta\phi(x, y)+\lambda[C(x^2+y^2)-2Gx-2Fy+A+B]+\lambda^2=0,$$

i.e., on putting $\lambda=k\Delta$, it is

$$\phi+k\psi+k^2\Delta=0.$$

It will be noted that $\psi=0$ is the equation to the director circle of the conic (Part I, Art. 390).

Cor. It may be deduced, by putting $\lambda=\mu c$, that all conics confocal with $ax^2+2hxy+by^2+c=0$ are given by

$$ax^2+2hxy+by^2+\mu(x^2+y^2)+c\frac{(a+\mu)(b+\mu)-h^2}{ab-h^2}=0,$$

where μ is some constant.

170. Ex. 1. *Find the equation to the conic, whose foci are the circum-centre and ortho-centre of a triangle ABC, and which touches the side BC.*

The circum-centre is $(\cos A, \cos B, \cos C)$ and the ortho-centre is $(\cos B\cos C, \cos C\cos A, \cos A\cos B)$.

Hence the tangential equation of the conic is

$$\begin{aligned}(l\cos A+m\cos B+n\cos C)(l\cos B\cos C+m\cos C\cos A+n\cos A\cos B)\\ =\lambda(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C).\end{aligned}$$

Since the conic touches BC this equation must be satisfied by $(1, 0, 0)$.

Hence $\lambda=\cos A\cos B\cos C$, and the equation becomes

$$mn\cos A(\cos^2 B+\cos^2 C+2\cos A\cos B\cos C)+\ldots+\ldots=0,$$

i.e. $mn\cos A(\cos B\sin A\sin C+\cos C\sin A\sin B)+\ldots+\ldots=0,$

i.e. $mn\cos A\sin^2 A+nl\cos B\sin^2 B+lm\cos C\sin^2 C=0.$

Hence the point equation is

$$\sin A\sqrt{\alpha\cos A}+\sin B\sqrt{\beta\cos B}+\sin C\sqrt{\gamma\cos C}=0.$$

The conic therefore touches the other two sides of the triangle, as would be expected, since the product of the perpendiculars on any side of the triangle from the circum- and ortho-centres is the same.

Ex. 2. *Shew that there are four, and only four, conics inscribed in a triangle each confocal with a conic circumscribed to the triangle, and find their equations.*

A circumscribed conic is $L\beta\gamma+M\gamma\alpha+N\alpha\beta=0$ and its tangential equation is $\sqrt{Ll}+\sqrt{Mm}+\sqrt{Nn}=0$, *i.e.*

$$L^2l^2+M^2m^2+N^2n^2-2MNmn-2NLnl-2LMlm=0.$$

The tangential equation of any confocal is

$$L^2l^2+M^2m^2+N^2n^2-2MNmn-2NLnl-2LMlm$$
$$+\lambda\,(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)=0.$$

If this conic is inscribed in the triangle of reference, this equation must be satisfied by

$$(1, 0, 0), (0, 1, 0), \text{ and } (0, 0, 1).$$

Hence $$L^2=M^2=N^2=-\lambda,$$

and the confocal is

$$mn\,(L^2\cos A-MN)+nl\,(M^2\cos B-NL)+lm\,(N^2\cos C-LM)=0.$$

The four different cases are then given by

$$L=M=N;\quad L=M=-N;\quad L=-M=N; \text{ and } -L=M=N.$$

The corresponding circum-conics are

$$\beta\gamma+\gamma\alpha+\alpha\beta=0;\quad \beta\gamma+\gamma\alpha-\alpha\beta=0;$$
$$\beta\gamma-\gamma\alpha+\alpha\beta=0, \text{ and } -\beta\gamma+\gamma\alpha+\alpha\beta=0.$$

The corresponding confocal in-conics have as tangential equations

$$mn\sin^2\frac{A}{2}+nl\sin^2\frac{B}{2}+lm\sin^2\frac{C}{2}=0;\quad mn\cos^2\frac{A}{2}+nl\cos^2\frac{B}{2}-lm\sin^2\frac{C}{2}=0;$$

$$mn\cos^2\frac{A}{2}-nl\sin^2\frac{B}{2}+lm\cos^2\frac{C}{2}=0;\quad -mn\sin^2\frac{A}{2}+nl\cos^2\frac{B}{2}+lm\cos^2\frac{C}{2}=0;$$

and as point equations

$$\sin\frac{A}{2}\sqrt{\alpha}+\sin\frac{B}{2}\sqrt{\beta}+\sin\frac{C}{2}\sqrt{\gamma}=0;$$

$$\cos\frac{A}{2}\sqrt{\alpha}+\cos\frac{B}{2}\sqrt{\beta}+\sin\frac{C}{2}\sqrt{-\gamma}=0;$$

$$\cos\frac{A}{2}\sqrt{\alpha}+\sin\frac{B}{2}\sqrt{-\beta}+\cos\frac{C}{2}\sqrt{\gamma}=0;$$

$$\sin\frac{A}{2}\sqrt{-\alpha}+\cos\frac{B}{2}\sqrt{\beta}+\cos\frac{C}{2}\sqrt{\gamma}=0;$$

EXAMPLES.

1. Find the foci of the conic whose tangential equation in Cartesian coordinates is

$$19l^2+16m^2+4lm-1=0.$$

2. From the tangential equation of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

shew that its foci are given by the equations

$$Cx^2-2Gx+A=Cy^2-2Fy+B=K,$$

where K is either root of the quadratic

$$CK^2-(a+b)\,\Delta k+\Delta^2=0.$$

3. Shew that the circum-centre of the triangle of reference is one focus of the conic whose tangential equation is

$$qr\sin 2A+rp\sin 2B+pq\sin 2C=0,$$

and find the other focus.

4. Shew that the trilinear equation to the ellipse through B and C which has one focus at the angular point A of the triangle of reference ABC, and the other focus in BC, is

$$\alpha^2\sin^2\frac{A}{2}+\beta\gamma+\gamma\alpha+\alpha\beta=0.$$

5. Shew that it is possible to draw three equal parabolas, with the ortho-centre of a triangle as focus, each touching two sides of the triangle.

6. Shew that the locus of the foci of the conic $\beta\gamma=k\alpha^2$, for different values of k, is the cubic curve

$$\alpha(\beta^2-\gamma^2)+2\beta\gamma(\beta\cos B-\gamma\cos C)=0.$$

7. The coordinates being areal, shew that the conics given by

$$\sqrt{ap}+\sqrt{bq}+\sqrt{cr}=0,$$

and

$$\tan\frac{A}{2}qr+\tan\frac{B}{2}rp+\tan\frac{C}{2}pq=0,$$

are confocal.

8. A conic, confocal with the conic whose tangential equation is $\Sigma=0$, touches the sides of the triangle ABC. Shew that, of the conics which touch the four tangents from the points B, C to Σ, one has a focus at A and the companion focus on BC.

9. The sides of a triangle inscribed in an ellipse touch a confocal ellipse; shew that the points of contact are the points at which the sides touch the corresponding escribed circles.

10. A triangle is self-conjugate with respect to a conic; shew that the sides of its pedal triangle touch a confocal conic.

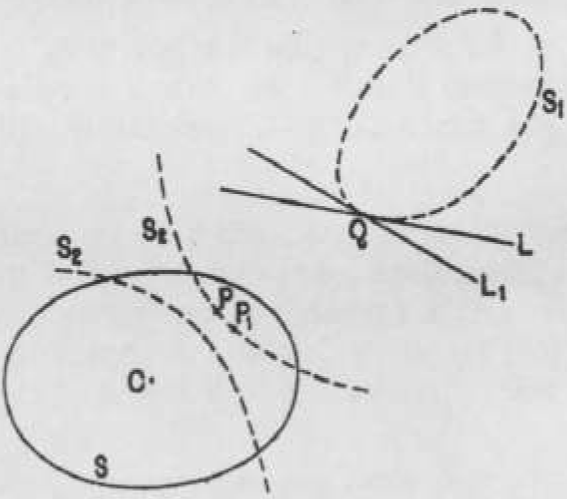
CHAPTER VI

RECIPROCAL POLARS

171. If for any line L we take its pole P with respect to a conic S, then, corresponding to any collection of lines such as L, we shall get a series of points P, and if the lines L touch a given curve the points P will lie on another curve.

So, if for any point P we take its polar L with respect to a conic S, then, corresponding to any series of points P we shall get a collection of lines L, and if the points P lie on a given curve the lines L will touch another curve.

Such a point P and a line L are said to correspond to one another.



Suppose that all the lines L touch a given curve S_1; let two of them, L and L_1, meet in Q, and let their poles with regard to the conic S be P and P_1. Since the polar of P is L, which passes through Q, therefore the polar of Q passes through P. So since L_1, the polar of P_1, passes through Q, the polar of Q passes through P_1.

Hence the polar of Q is the line PP_1.

Now let L_1 very nearly coincide with L, so that the point

Q very nearly lies on the curve S_1; then the points P_1 and P very nearly coincide also, and are very close points to one another on the curve S_2 on which all the points $P, P_1, P_2, \ldots$ lie.

Hence, in the limit, when Q actually lies on the curve S_1, the line PP_1 becomes a tangent to the curve S_2.

Hence if, instead of starting with tangents $L, L_1, \ldots$ to S_1 and so obtaining the points $P, P_1, \ldots$ on S_2, we started with points such as Q on S_1 and took their polars, we should obtain tangents to the curve S_2.

Thus whether we started with tangents to, or points on, S_1, and took their poles, or polars, with respect to S, we should obtain the same curve S_2, either as the loci of these poles, or as the envelope of these polars.

So similarly whether we started with tangents to, or points on S_2, and took their poles, or polars, with respect to S, we should obtain the same curve S_1, either as the loci of these poles, or as the envelope of these polars.

On account of this property either curve, S_1 or S_2, is said to be the **Reciprocal Polar** of the other with regard to the conic S, and this conic S is called the **Auxiliary Conic.**

172. To any tangent L to S_1, which does not cut the conic S in real points, there corresponds a point P within S; if L cuts S in real points the corresponding point P is without S; if the tangent L is also a tangent to S, the corresponding point P is on S, and is the point of contact of L with S.

If from any point Q there can be drawn two, three, ... tangents to S_1, then the corresponding polar of Q with respect to S cuts S_2 in two, three, ... points, each one of these latter points being the pole with respect to S of one of the tangents from Q to S_1.

Hence the number of points in which any given line cuts S_2 is the same as the number of tangents that can be drawn to S_1 from the point corresponding to that given line, *i.e.* the degree of S_2 is the same as the class of S_1.

Similarly the class of S_2 is the same as the degree of S_1.

In particular, if S_1 be a conic (which is a curve of degree 2 and class 2) then S_2 will be a curve of class 2 and degree 2, so that S_2 will also be a conic.

Similarly, if S_2 is a conic, so also will S_1 be.

Thus *The polar reciprocal of a conic with respect to a conic is also a conic.*

173. If from the centre C of the auxiliary conic S we can draw two real tangents to S_1, then corresponding to these two real tangents we shall have two real points at infinity, and S_2 will be a hyperbola. For the pole with respect to S of any line through its centre C is at infinity.

So, if we can only draw one tangent from C to S_1 (*i.e.* if C lie on S_1), we shall get one real point at infinity, and S_2 will be a parabola.

If we can draw no real tangents from C to S_1 (*i.e.* if C is within S_1), we shall get no real points at infinity, and S_2 will be an ellipse.

Hence *The reciprocal polar of any conic S_1 with regard to a conic S will be an ellipse, parabola, or hyperbola according as the centre of S is within, upon, or without the conic S_1.*

174. We observe that if S_1, S_2 are the polar reciprocals of one another with respect to a conic S there is a correspondence of lines and points. Thus:

To a point (or line) of one curve corresponds a line (or point) of the other curve.

To a point on (or a tangent to) one curve corresponds a tangent to (or a point on) the other curve.

To the point of contact of a tangent to one curve corresponds a tangent at the corresponding point on the other curve.

To the line joining two points on one curve corresponds the point of intersection of two tangents to the other curve.

To the line joining the points of contact of two tangents to one curve corresponds the point of intersection of the tangents at the two corresponding points of the other curve.

To the point of contact, *i.e.* two coincident points of intersection, of two curves corresponds two coincident tangents on the two corresponding curves. Hence if two curves touch, their reciprocals also touch; also, if two conics have double contact, their reciprocals have double contact also.

To a self-conjugate triangle with respect to a conic (*i.e.* a triangle, the intersection of two of whose sides is the pole of the third side) corresponds a triangle, the line joining two of whose angular points is the polar of the third angular point, *i.e.* a self-conjugate triangle with respect to the new conic.

175. After a little practice the student will find it easy to write down the theorem which is the reciprocal of a given theorem. He will find it to consist simply in writing "straight line" for "point," "intersect" for "join," "polar" for "pole," "locus" for "envelope," "tangent to" for "point on," "lie on a straight line" for "meet in a point," "point of contact of a tangent" for "tangent at a given point," etc., and *vice versâ*.

Some examples are now given. In one column is given a theorem and in a parallel column the reciprocal theorem. The student should take any one of these theorems, and, without looking at the reciprocal theorem, write the latter down for himself.

Given four points on a conic, the polar of a fixed point passes through a second fixed point.	Given four tangents to a conic, the pole of a fixed straight line lies on a second fixed straight line.
Given four points on a conic, the locus of the pole of a fixed straight line is a conic section.	Given four tangents to a conic, the envelope of the polar of a fixed point is a conic section.
If a hexagon be inscribed in a conic, the intersections of the opposite sides lie on a straight line. (Pascal's Theorem)	If a hexagon be circumscribed to a conic, the lines joining its opposite angular points meet in a point. (Brianchon's Theorem)

If two triangles are inscribed in a conic, their six sides touch another conic.	If two triangles are circumscribed to a conic, then six angular points lie on another conic.
If two triangles are self-conjugate with regard to a conic, their six angular points lie on another conic.	If two triangles are self-conjugate with regard to a conic, their six sides touch another conic.
The three lines joining each vertex of a triangle to the pole of its opposite side with regard to a conic are concurrent.	The three points of intersection of each side of a triangle with the polar of the opposite angular point with regard to a conic are collinear.
A triangle is inscribed in a conic; the points in which each side meets the tangent at the opposite angular point are collinear.	A triangle is circumscribed to a conic; the straight lines joining each angular point with the point of contact of the opposite side are concurrent.
Two angular points of a triangle lie on two fixed tangents to a conic, the side joining them is a tangent to the conic, and each of the remaining sides passes through a fixed point; the locus of the third angular point of the triangle is a conic passing through the two fixed points.	Two sides of a triangle pass through two fixed points of a conic, their intersection is a point on the conic, and each of the remaining angular points lies on a fixed straight line; the envelope of the third side of the triangle is a conic which touches the two fixed straight lines.

176. Ex. 1. *A straight line is drawn to be cut harmonically by two given circles. Its envelope is a conic whose foci are the centres of the two circles* (Page 11, Ex. 2).

Rewrite. A straight line cuts two circles (which pass through the same two points on the line at infinity) in points giving a harmonic range. Its envelope is a conic inscribed in the quadrilateral, whose vertices are the poles of the line at infinity with respect to the two circles, and the intersections of the circles with the line at infinity.

Reciprocate. A point is taken such that the tangents from it to two conics, which touch two straight lines passing through the centre C of the auxiliary conic, form a harmonic pencil. Its locus is a conic circumscribing the quadrangle, whose sides are the polars of C with respect to the two conics and the tangents from C to the two conics.

Hence, *The locus of points from which the tangents to two given conics form a harmonic pencil is a conic passing through the four points of contact of a pair of common tangents* (Arts. 113 and 117).

Obtain the reciprocals of the following theorems:

Ex. 2. Triangles in perspective are coaxal.

Ex. 3. If a conic is inscribed in a triangle, the straight lines joining each angular point to the point of contact with the opposite side are concurrent.

Ex. 4. If a conic circumscribes a quadrangle, it is self-conjugate with respect to the triangle formed by the diagonal points of the quadrangle.

Ex. 5. If a triangle is circumscribed to a conic, and two of its vertices lie on given straight lines, the locus of the third vertex is a conic having double contact with the given conic.

177. From this transformation by reciprocal polars it appears that to every theorem about points and lines there corresponds a theorem about lines and points. This is known as the **Principle of Duality.**

178. *To shew that the polar reciprocal of one conic S_1 with regard to an auxiliary conic S is another conic having a common self-conjugate triangle with S and S_1.*

Take as triangle of reference the common self-conjugate triangle of S and S_1, so that

$$S \equiv a\alpha^2 + b\beta^2 + c\gamma^2 = 0,$$

and

$$S_1 \equiv a_1\alpha^2 + b_1\beta^2 + c_1\gamma^2 = 0.$$

We want the locus of the point $(\alpha', \beta', \gamma')$ such that its polar, viz. $a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' = 0$, with respect to S shall always touch $S_1 = 0$. The condition for this is found by substituting $(a\alpha', b\beta', c\gamma')$ for l, m, n in the tangential equation of S_1 (Art. 72), and the locus of $(\alpha', \beta', \gamma')$ is thus seen to be

$$S_2 \equiv \frac{a^2}{a_1}\alpha^2 + \frac{b^2}{b_1}\beta^2 + \frac{c^2}{c_1}\gamma^2 = 0,$$

a conic having a common self-conjugate triangle with S and S_1.

Similarly the reciprocal polar of S_2 with regard to S is

$$\frac{a^2}{\frac{a^2}{a_1}}\alpha^2+\frac{b^2}{\frac{b^2}{b_1}}\beta^2+\frac{c^2}{\frac{c^2}{c_1}}\gamma^2=0,$$

i.e. it is $$S_1 \equiv a_1\alpha^2+b_1\beta^2+c_1\gamma^2=0.$$

179. *To find the polar reciprocal of the conic*

$$\phi_1(\alpha, \beta, \gamma) \equiv a_1\alpha^2+b_1\beta^2+c_1\gamma^2+2f_1\beta\gamma+2g_1\gamma\alpha+2h_1\alpha\beta=0$$

with respect to the conic

$$\phi(\alpha, \beta, \gamma) \equiv a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0.$$

We want the locus of the point $(\alpha', \beta', \gamma')$ which is such that its polar with regard to $\phi=0$ always touches $\phi_1=0$. This polar is

$$\alpha(a\alpha'+h\beta'+g\gamma')+\beta(h\alpha'+b\beta'+f\gamma')+\gamma(g\alpha'+f\beta'+c\gamma')=0,$$

and the condition that it shall touch $\phi_1=0$ is obtained by substituting $a\alpha'+h\beta'+g\gamma'$, $h\alpha'+b\beta'+f\gamma'$, and $g\alpha'+f\beta'+c\gamma'$ for l, m, n in the tangential equation of ϕ_1. We thus have for the locus of $(\alpha', \beta', \gamma')$,

$$\begin{aligned}A_1(a\alpha+h\beta+g\gamma)^2+B_1(h\alpha+b\beta+f\gamma)^2+C_1(g\alpha+f\beta+c\gamma)^2&\\+2F_1(h\alpha+b\beta+f\gamma)(g\alpha+f\beta+c\gamma)&\\+2G_1(g\alpha+f\beta+c\gamma)(a\alpha+h\beta+g\gamma)&\\+2H_1(a\alpha+h\beta+g\gamma)(h\alpha+b\beta+f\gamma)&=0,\end{aligned}$$

where A_1, B_1, C_1, ... are derived from the coefficients of ϕ_1 in the usual way.

EXAMPLES.

1. Shew that a hyperbola is its own reciprocal with respect to its conjugate hyperbola.

2. Shew that the polar reciprocal of the conic $\beta\gamma=k\alpha^2$ with regard to the conic $\beta\gamma=k_1\alpha^2$ is the conic

$$\beta\gamma=\frac{k'^2}{k}\alpha^2.$$

3. Shew that the reciprocal of a circle with respect to a rectangular hyperbola is a conic whose focus is at the centre of the hyperbola, and that the reciprocal of a rectangular hyperbola with respect to another concentric rectangular hyperbola is also a rectangular hyperbola.

4. If a conic $S=0$ is its own polar reciprocal with respect to a conic $S'=0$, then S' is its own polar reciprocal with respect to S.

5. Shew that the three conics

$$x^2+y^2-2cx\sqrt{3}-c^2=0, \quad x^2+y^2+2cx\sqrt{3}-c^2=0,$$

and

$$2x^2-y^2-2c^2=0,$$

are such that any one is the reciprocal of the second with regard to the third when they are taken in any order.

6. Shew that the conic

$$2(l\lambda x+m\mu y+n\nu z)^2=(l^2+m^2+n^2)(\lambda^2x^2+\mu^2y^2+\nu^2z^2)$$

is its own polar reciprocal with respect to the conic

$$\lambda^2x^2+\mu^2y^2+\nu^2z^2=0.$$

7. Two conics are such that each is its polar reciprocal with respect to the other. Shew that they have double contact with one another, and that the tangents drawn to them from any point on the chord of contact form a harmonic pencil.

8. Shew that a conic is its own polar reciprocal with respect to four conics, of which three are real and the fourth is imaginary.

9. If $U=0$, $V=0$, and $W=0$ are the equations of three conics such that one is the reciprocal polar of the second with respect to the third, prove that they have a common self-conjugate triangle, the equation to whose sides is $J=0$, where J is the Jacobian of U, V, W. [See Page 94, Ex. 15.]

180. The conic S with regard to which the reciprocation is made is in practice nearly always a circle, and in the remainder of this book we shall assume it to be always a circle unless the contrary is stated.

It has been shewn in Part I, Art. 165, that the pole of any line L with regard to a circle of centre O is obtained by drawing ON perpendicular to L, and on it taking a point P such that $OP \cdot ON =$ the square of the radius of the circle.

Hence the polar reciprocal of any curve with regard to a circle, centre O, is obtained by drawing a perpendicular ON upon any tangent to the curve, and on it taking a point P such that $OP=\frac{k^2}{ON}$, where k is the radius of the auxiliary circle.

Hence, if we have two tangents, L and L', to the conic and their poles, P and P', with respect to the auxiliary circle, the angle between the two lines L and L' is equal to the angle POP'.

Hence *The angle between any two tangents to the original conic is equal to the angle subtended at the origin by the two corresponding points on the reciprocal curve, and conversely.*

181. *To find the polar reciprocal of the general conic*

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots(1)$$

with regard to a circle, whose radius is k, and whose centre is the origin, O, of coordinates.

We want the locus of a point (x', y') whose polar with respect to the circle, viz. $xx' + yy' - k^2 = 0$, touches the conic $\phi = 0$. The condition for this is found by substituting x', y' and $-k^2$ for l, m and n in the tangential equation of ϕ (Art. 72).

The locus of (x', y') is thus

$$Ax^2 + 2Hxy + By^2 - 2k^2Gx - 2k^2Fy + Ck^4 = 0 \ldots(2),$$

where $A, B, C, \ldots$ are found from the coefficients of ϕ in the usual way.

By Part I, Art. 389, the equation to the pair of tangents from the origin to the original conic is

$$Bx^2 - 2Hxy + Ay^2 = 0 \ \ldots\ldots\ldots\ldots\ldots(3),$$

and the asymptotes of the reciprocal conic (2) are parallel to

$$Ax^2 + 2Hxy + By^2 = 0 \ \ldots\ldots\ldots\ldots\ldots(4).$$

The straight lines (4) are clearly perpendicular to the straight lines (3).

Therefore the asymptotes to the reciprocal conic are perpendicular to the tangents from the origin to the original conic, and hence the angle between these asymptotes is supplementary to the angle between the tangents to the original conic.

182. This also appears geometrically. For if we draw a tangent OT, or OT', from O to the original conic the perpendicular, ON, on it from O vanishes, and hence the corresponding point P is at an infinite distance, since $OP = \frac{k^2}{ON}$. Also the direction of OP is perpendicular to the tangent OT, and, since P is at an infinite distance, OP is parallel to an asymptote. Hence the asymptotes are perpendicular to OT and OT', and, as before, the angle between them is supplementary to the angle TOT'.

Conversely, the points at infinity on the original conic correspond to the tangents through the origin to the reciprocal conic, and hence the angle between the asymptotes of the original conic is equal or supplementary to the angle between the tangents from the origin to the reciprocal conic.

The asymptotes of the reciprocal conic are imaginary, coincident, or real, according as the tangents from the origin O to the original conic are imaginary, coincident, or real, *i.e.* according as O lies within, upon, or without the original conic.

183. If the origin O is a point on the director circle of the original conic, the tangents from it are at right angles, so that the asymptotes of the reciprocal conic are at right angles. Hence, if we reciprocate a conic with respect to a point on its director circle, we obtain a rectangular hyperbola, and conversely, if we reciprocate a rectangular hyperbola with respect to a point O, we have a conic whose director circle passes through O.

184. Ex. 1. The reciprocal of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the origin is the conic $a^2x^2+b^2y^2=k^4$, a coaxal conic whose axes are the reciprocals of the original conic.

Ex. 2. The reciprocal of the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to any point (x', y') is

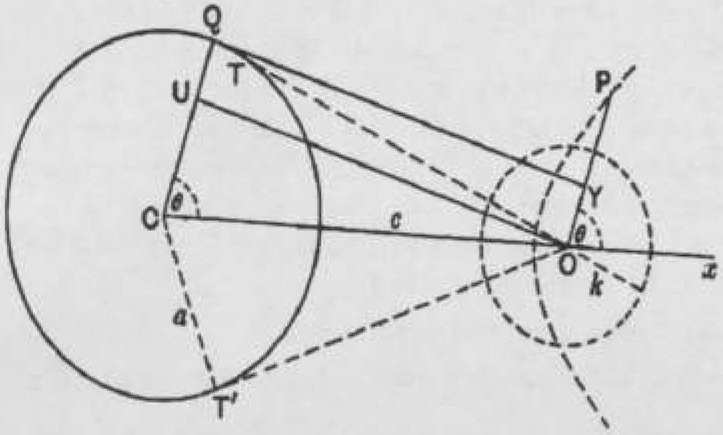
$$a^2(x-x')^2+b^2(y-y')^2=[x'(x-x')+y'(y-y')+k^2]^2.$$

185. Polar reciprocal of one circle with respect to another.

Let C be the centre of the circle to be reciprocated and a its radius. Let O be the centre of the auxiliary circle, and k its radius. Draw OY perpendicular to any tangent at Q to the circle of centre C, and let P be a point on OY such that $OP \cdot OY=k^2$. Then the locus of P is required.

Draw OU perpendicular to CQ, and let

$$\angle QCO = \angle POx = \theta, \text{ and } OP = r.$$



Then $a = CU + OY = CO\cos\theta + \dfrac{k^2}{OP} = c\cos\theta + \dfrac{k^2}{r}$.

$$\therefore\ r = \frac{\dfrac{k^2}{a}}{1 - \dfrac{c}{a}\cos\theta}.$$

Hence (Part I, Art. 335), the locus of P is a conic section, whose focus is O, whose semi-latus rectum is $\dfrac{k^2}{a}$, and whose eccentricity is $\dfrac{c}{a}$.

It is therefore an ellipse, parabola, or hyperbola according as $c \lesseqgtr a$, *i.e.* according as the origin O lies within, upon, or without the circle that is reciprocated.

As in Part I, Art. 339, the directrix is

$$x = r\cos\theta = -\frac{l}{e} = -\frac{k^2}{c} = -\frac{k^2}{OC},$$

and hence the directrix of the reciprocal conic is the polar of the centre, C, of the original circle with respect to the auxiliary circle. Hence *The centre of the given circle reciprocates into the directrix of the reciprocal conic.*

Ex. If $2a_1$, $2b_1$ are the major and minor axes of the reciprocal conic, then

$$a_1 = \frac{k^2 a}{a^2 - c^2} \text{ and } b_1 = \frac{k^2}{\sqrt{a^2 - c^2}}.$$

186. If O be without the given circle, the tangents to the latter at two points, T and T', pass through O. For the tangent TO the perpendicular from O vanishes, and the corresponding point P is at an infinite distance, and lies on a line through O perpendicular to OT. So for the tangent $T'O$. Hence the lines through the focus O, parallel to the asymptotes, are perpendicular to OT and OT' respectively, and hence the angle between the two asymptotes is equal to the supplement of the angle between the tangents that can be drawn to the original circle from O.

This is also clear from the equation.

For the angle that an asymptote makes with the axis Ox

$$=\cos^{-1}\frac{1}{e} \text{ (Part I, Art. 315)} = \cos^{-1}\frac{a}{c} = \angle TCO$$

$$=\frac{\pi}{2} - \angle TOC.$$

$\therefore$ the required angle between the asymptotes

$$=\pi - 2\angle TOC.$$

187. To any four points lying on a straight line correspond four straight lines meeting in a point, and the cross-ratio of the four points is equal to that of the pencil of the four straight lines.

For if the four points be P, Q, R, S and their reciprocals with respect to an origin O be the straight lines P_1, Q_1, R_1, S_1, the angle between the straight lines P_1 and Q_1 is equal to the angle POQ (Art. 180), and so for the others. Hence, if the four lines meet in the point O_1, it is clear that

$$[PQRS] = O[PQRS] = O_1[P_1Q_1R_1S_1].$$

Hence we have the important theorems of Art. 119 by reciprocation from the simple case of the circle.

The cross-ratio of the pencil joining four points of a circle to any fifth point is constant.	The cross-ratio of the points of intersection of four tangents to a conic with any fifth tangent is constant.
The cross-ratio of the points of intersection of four tangents to a circle with any fifth tangent is constant.	The cross-ratio of the pencil joining four points of a conic to any fifth point is constant.

The first theorem on the left is clearly true, since the angles at the fifth point are unaltered as it moves.

The second theorem is true, because the angle that the line joining any two points of the four subtends at the centre of the circle is constant, being equal to one-half the angle subtended there by the corresponding points of contact.

188. *To shew that a system of four-point conics can be reciprocated into a system of concentric conics, and that a system of four-line conics can be reciprocated into rectangular hyperbolas and also into concentric conics.*

For the four-point conics take as origin of reciprocation one of the angular points of the common self-conjugate triangle. Then the polar of the origin is the same for each conic, being the opposite side of the self-conjugate triangle. Hence, in the reciprocal system of conics, the pole of the line at infinity is the same for each, *i.e.* they have a common centre.

For the four-line conics we know (Art. 99, Ex. 6) that their director circles all intersect in the same two points. Take one of these two points as origin. Then the tangents from the origin to each conic of the system are at right angles. Hence in the reciprocal system the asymptotes are at right angles, and the reciprocal conics are therefore all rectangular hyperbolas.

As in the first part, if we take as origin of reciprocation an angular point of the common self-conjugate triangle (in this case the triangle formed by the diagonals), we obtain a system of concentric conics.

189. Ex. 1. *The angles in the same segment of a circle are equal.*

This may be written thus: If Q and R be fixed points on a circle, and P any other point on the circle, the angle between the lines PQ and PR is constant.

Reciprocate with respect to any point O and we have the following proposition: If Q and R are fixed tangents to a conic, of focus O, and if P be any other tangent to the conic, then the angle subtended at O by the points PQ and PR is constant, *i.e.*

If a variable tangent to a conic cut two fixed tangents in A and B, the angle subtended by AB at a focus is constant.

Ex. 2. *In any triangle the perpendiculars from the vertices upon the opposite sides meet in a point.*

Rewrite. If A, B, C are the angular points of a triangle, and if on BC is taken a point D such that the lines DC and DA are at right angles, and similarly for E and F, then AD, BE and CF meet in a point.

Reciprocate with respect to a point O. If A, B, C are the sides of a triangle, and if through BC is drawn a line D such that the points DC and DA subtend a right angle at O, and similarly for E and F, then the points AD, BE and CF lie on a straight line.

Hence, *If PQR be a triangle, and if through a point O be drawn lines* OP_1, OQ_1, OR_1 *perpendicular to* OP, OQ, OR *to meet* QR, RP, *and* PQ *in* P_1, Q_1, R_1, *then* P_1, Q_1, R_1 *are collinear.*

Ex. 3. *The locus of the centre of circles touching two straight lines is the bisectors of the angles between the two given lines.*

Rewrite. Circles touch two given straight lines P and Q; the locus of the pole of the line at infinity with respect to these circles is one or other of the lines, R, which pass through the intersection of P and Q, and are such that the angle between P and R is equal or supplementary to that between Q and R.

Reciprocate with respect to a point O. Conics, of focus O, pass through two given points P and Q; the envelope of the polar of the origin O with respect to these conics is one or other of the points R which lie on the line PQ, such that the angle that P and R subtends at O is equal or supplementary to that subtended by Q and R.

Hence, *If a conic whose focus is O passes through two points P and Q, its directrix passes through one of the points on PQ where it is met by the internal or external bisector of the angle POQ.*

Ex. 4. *The rectangle contained by the segments of any chord drawn through a fixed point O to meet a circle is constant.*

Let the chord meet the circle in P and Q; then P, Q being points lying on a line passing through O will reciprocate, with O as centre, into tangents which meet on the line at infinity, *i.e.* which are parallel. Also the perpendicular on the tangent corresponding to P is $\dfrac{k^2}{OP}$.

Hence *If from the focus O of a conic perpendiculars be drawn to two parallel tangents, their product is constant.*

Ex. 5. *The locus of the intersection of perpendicular tangents to a conic is a concentric circle; also if the conic is one of a system inscribed in a given quadrilateral, the corresponding circles are coaxal.*

Reciprocate with respect to a point O. The envelope of the line joining two points on a given conic, which subtend a right angle at O, is a conic, of focus O, such that the polar of O with respect to it and the conic is the same; also, if the given conic is one of a system passing through four given points, the corresponding loci touch the same two straight lines.

Ex. 6. *A system of conics have one focus and a pair of tangents common; the corresponding directrices pass through a common point, and have all their centres on the same straight line.*

Rewrite. A system of conics have one focus S and a pair of tangents common; the polars of S pass through a common point and the poles of the line at infinity lie on the same straight line.

Reciprocate with respect to S. A system of circles pass through two common points; the poles of the line at infinity lie on a line, and the polars of the origin S meet in a point.

Hence, *In a system of coaxal circles the centres are collinear, and the polars of a fixed point meet in another fixed point.*

190. Some further examples follow:

Four circles can be drawn to touch the sides of a given triangle, and the sum of the reciprocals of the radii of three of them is equal to the reciprocal of the radius of the fourth. Also the lines joining their centres pass, two by two, through the angular points of the triangle.	With a given point as focus four conics can be drawn to circumscribe a given triangle, and the sum of the latera recta of three of them is equal to the latus rectum of the fourth. Also their directrices meet, two by two, on the sides of the triangle.
If from a point on the circumcircle of a triangle perpendiculars be drawn to the sides, their feet are collinear.	ABC is a triangle whose sides touch a conic, of focus O; the perpendiculars to OA, OB, OC passing through O meet any tangent in A_1, B_1, C_1; then AA_1, BB_1, CC_1 meet in a point.
The locus of the centres of all conics inscribed in a given quadrilateral is a straight line.	The envelope of the polars of a point O with regard to a system of conics circumscribing a given quadrilateral is a point, *i.e.* the polars of any point all pass through another point.
If two conics have a common focus, a point of intersection of their directrices and a point of intersection of their common tangents subtend a right angle at the focus.	[Reciprocate with respect to the focus.] The line joining the centres of two circles, and the line joining their common points, are perpendicular.
The ortho-centre of a triangle circumscribing a parabola lies on its directrix.	[Reciprocate with respect to the ortho-centre.] The ortho-centre of a triangle inscribed in a rectangular hyperbola lies on the hyperbola.

The sum of the distances from the focus of an ellipse of the points of contact of any two parallel tangents is constant.	If from a point within a circle perpendiculars be let fall upon any two tangents, whose chord of contact passes through the point, the sum of their reciprocals is constant.

191. Reciprocation of a system of coaxal circles.

With the figure of Part I, Art. 190, it can be easily shewn that the polar of either limiting point L_1, with respect to any circle of the coaxal system, passes through the other limiting point.

For any circle of the system is, by Art. 188,

$$x^2 + y^2 - 2gx + c = 0,$$

and L_1 is the point $(-\sqrt{c}, 0)$.

Hence the polar of L_1 is

$$-x\sqrt{c} - g(x - \sqrt{c}) + c = 0,$$

i.e. $$(x - \sqrt{c})(g + \sqrt{c}) = 0,$$

i.e. it is a straight line through the other limiting point L_2, parallel to the common radical axis of the system.

If then we reciprocate the system of circles with respect to the limiting point L_1, we have a system of conics, one of whose foci is L_1; also, since the polar of the origin with respect to the circles is the same for all, the pole of the line at infinity is the same for all the reciprocal conics, *i.e.* the centre of the reciprocal conics is the same for all.

Hence the reciprocal conics have one focus, L_1, the same for all, and they all have the same centre; hence the second focus is the same for all, and the reciprocal conics are confocal.

In the reciprocal figure the second focus is twice as far from the origin L_1 as the centre is, and hence the reciprocal of the second focus is one-half as far from L_1 as the reciprocal of the centre, *i.e.* it is one-half as far from L_1 that L_2 is.

Hence the reciprocal of the second focus is the radical axis OY of the system of coaxal circles.

Hence, *If we reciprocate a system of coaxal circles with regard to one of the limiting points L_1, we obtain a system of* *confocal conics, one of whose foci is L_1, and whose second focus and centre are respectively the reciprocals of the common radical axis and the line through the other limiting point L_2 perpendicular to L_1L_2, and conversely.*

192. Ex. 1. *Confocal conics cut at right angles.*

Rewrite. A system of confocal conics is drawn; the tangents at a common point of any two of them are at right angles.

Reciprocate with respect to a focus S. If a system of coaxal circles be drawn, one of whose limiting points is S, then the points of contact of a common tangent with two of the circles of the system subtend a right angle at S.

Ex. 2. *The tangents to two confocal conics at a point of intersection bisect the angles between the focal distances of the point.*

Rewrite. A system of conics have two points S_1 and S_2 as foci, and P is a point of intersection of two of these conics; the two tangents at P and the lines PS_1, PS_2 form a harmonic pencil.

Reciprocate with respect to S_1. Then, remembering that the reciprocal of the point S_1 is the line at infinity, and that the reciprocal of the point S_2 is the radical axis of the reciprocal coaxal system, we have: P is a common tangent to two circles of a coaxal system; the two points of contact of a common tangent P, and the two points of intersection of P with the line at infinity and the radical axis, form a harmonic range.

Or, *A common tangent to two circles of a coaxal system is bisected by the radical axis.*

Ex. 3. *Two points are taken on two circles of a coaxal system which subtend a right angle at a limiting point L of the system; the envelope of the line joining them is a conic with focus L.*

Reciprocate. If tangents at right angles are drawn from a point P, one to each of two confocals, the locus of P is a circle.

Ex. 4. *The polar of a fixed point P with respect to a system of coaxal circles passes through a fixed point Q, the two points subtend a right angle at each of the limiting points of the system, and the line joining them is a tangent to that circle of the system which passes through P.*

Reciprocate with respect to one of the limiting points L_1. The pole of a fixed straight line P with respect to a system of confocal conics lies on a fixed straight line Q, the two straight lines are at right angles, and their point of intersection lies on that one of the confocals which touches the line P.

Hence, *The focus of the pole of a given straight line with respect to a system of confocals is the normal to that confocal which touches the given straight line.*

EXAMPLES.

Obtain more general theorems by reciprocating the following propositions:

1. The two opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

2. The tangent at any point of a circle is perpendicular to the radius to its point of contact.

3. The angle between a tangent at any point of a circle and any chord through its point of contact is equal to the angle in the alternate segment.

4. Every chord of a circle which subtends a right angle at a fixed point of the circle passes through the centre of the circle.

5. The locus of the intersection of tangents to a given circle which meet at a given angle, and the envelope of the line joining their points of contact, are both circles concentric with the given one.

6. The orthogonal trajectory of a system of coaxal circles is another system of coaxal circles passing through the limiting points of the first system.

7. The circle described on a focal radius of a parabola as diameter touches the tangent at the vertex.

8. The sum of the perpendiculars from any point within a circle upon two parallel tangents is constant.

Obtain the corresponding theorems of the following:

9. If a conic pass through two given points and touch two given lines, the line joining the points of contact of the two tangents passes through one or other of two fixed points.

10. If two vertices of a triangle move on fixed straight lines, and the sides pass each through a fixed point, the locus of the third vertex is a conic.

11. If three conics have each double contact with a fourth, their six chords of intersection will pass three by three through the same points.

12. Given two homographic ranges $P, Q, R, \ldots$ and $P', Q', R', \ldots$ on two straight lines; the envelope of the line PP' is a conic which touches the axes of the two ranges.

13. The circle circumscribing any triangle self-conjugate with regard to a rectangular hyperbola passes through its centre.

14. The tangents from any point to two confocal conics are equally inclined to each other.

15. The locus of the foot of the perpendicular from the focus of a conic upon any tangent to the conic is a circle concentric with the conic. [Reciprocate with respect to the focus.]

By reciprocation, prove the theorems:

16. The envelope of the chord of a conic which subtends a right angle at its centre is a concentric circle.

[Reciprocate with regard to the centre.]

17. Given three tangents to a parabola, the locus of its focus is the circum-circle of the triangle.

18. If a fixed line intersect a series of conics having the same focus and directrix, the envelope of the tangents to the conics at the points where this line meets them will be a conic, having the same focus, and touching both the fixed line and the common directrix.

19. Two conics having a common focus whose latera recta are $2l_1$, $2l_2$, and whose eccentricities are e_1, e_2, can have triangles inscribed in the first which are circumscribed to the second if

$$l_1^2 \pm 2l_1 l_2 = l_1^2 e_2^2 + l_2^2 e_1^2 - 2l_1 l_2 e_1 e_2 \cos \alpha,$$

where α is the angle between their axes.

[Reciprocate the theorem that the square of the distance between the centre of the circum-circle of a triangle and that of any circle touching the sides is $R^2 \pm 2Rr$.]

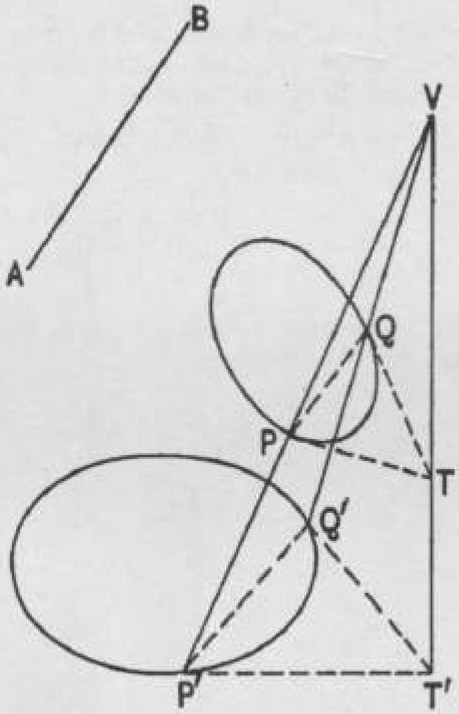
20. The normal at any point of a conic bisects the angle between the tangents from that point to a confocal conic.

CHAPTER VII

PROJECTION

193. If all the points of any plane figure are joined to any fixed point in space V, and the joining lines meet any other plane, they will give a series of points which are called the projections of the first series of points.

The point V is called the **vertex, or centre of projection,** and the second, or cutting, plane is called the **plane of projection.**



Thus, for every point P on the first figure, we have a point P' on the second figure, so that the projection of a point is always a point. Also a straight line is always projected into a straight line. For the lines joining V to any point on the line PQ in the first figure lie in the plane VPQ, passing through the vertex V and the line PQ, and this plane is cut by the plane of projection in another line $P'Q'$.

194. *A curve is always projected into a curve of the same degree.*

For if the original curve is cut by any straight line in the points P, Q, R, S, ... the projection of the curve will meet the lines VP, VQ, VR, VS, ... in the points P', Q', R', S', ... Hence the projection of any straight line will meet the projection of the curve in the same number of points in which the original straight line met the original curve, *i.e.* a curve and its projection are of the same degree.

In particular, *The projection of a conic section is always a conic section.*

195. *A tangent to a curve projects into a tangent to the projected curve.*

For, if the line QT passes through two coincident points at Q, the corresponding straight line $Q'T'$ will pass through two coincident points at Q'.

So if, in the original figure, we had two curves touching at Q, we shall, in the projected figure, have two curves touching one another at the point which is the projection of Q.

196. *The relation of pole and polar is unaltered by projection.*

For if, in the first figure, we have the tangents at P and Q intersecting in T, we shall, in the second figure, have the tangents at the corresponding points P' and Q' intersecting in the corresponding point T'.

197. If through the vertex V we draw a plane VAB *which is parallel to the plane of projection* to meet the original plane in the straight line AB, then the projection of any point C on AB will be at an infinite distance. For, VC being parallel to the plane of projection, the corresponding point C' will be at an infinite distance. Thus the straight line AB projects into a straight line at infinity, and hence is called the **Vanishing Line** of the original figure.

Similarly, if we draw a plane through V parallel to the original plane to meet the plane of projection in a line, then this line is the Vanishing Line of the projected figure. For this is the line on the plane of the projected figure which corresponds to the line at infinity on the plane of the original figure.

To project any given line AB of the original figure to infinity, it is therefore only necessary to take as the plane of projection any plane which is parallel to the plane through the vertex V and AB.

Again, any straight lines which meet at a point on AB will project into parallel lines; for their point of intersection will project into a point at infinity, and hence their projections will be parallel lines.

198. Similarly, parallel lines on the original plane will project into lines meeting in a point. For let a line through the vertex V, parallel to the system of parallel lines, meet the plane of projection in D. Then the plane through V and any one of the parallel lines will contain the line VD, and hence the intersection of this plane and the plane of projection will pass through the point D. The projection of any one of the parallel lines will thus pass through D, and hence their projections will be a set of lines concurrent in the point D.

Since VD is always parallel to the original plane, whatever be the directions of different sets of parallel lines, therefore D always lies on the intersection of the plane of projection and a plane through V parallel to the original plane, *i.e.* D always lies on a definite line in the plane of projection, whatever be the directions of the different sets of parallel lines in the original figures.

199. *A range of three points can be projected into any other range of three points in space.*

A range of four points can be projected into any other range of four points in space, provided that the cross-ratios of the two sets of four points are the same.

Let P, Q, R and P', Q', R' be the two ranges of three points which need not be in the same plane.

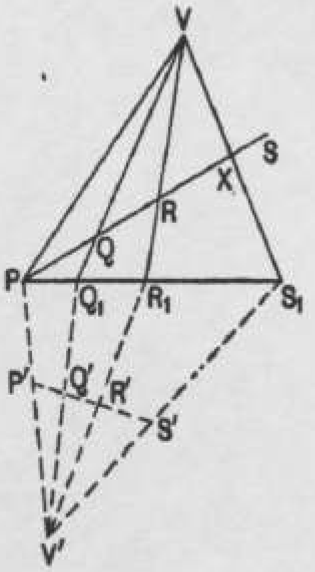
Take any point V' on the line joining P, P' and let PQ_1R_1 be any straight line in the plane through V' and the line $P'Q'R'$. Let $V'Q'$ and $V'R'$ meet this line in Q_1 and R_1.

Join Q_1Q, R_1R, and let them meet in V.

Then from the vertex V' the points P', Q', R' are projected into the points P, Q_1, R_1, and from the vertex V the points P, Q_1, R_1 are projected into the points P, Q, R.

Hence the first part of the proposition is proved.

For the second part let P, Q, R, S and P', Q', R', S' be the two ranges.



Construct as before, let $V'S'$ cut PQ_1R_1 in S_1, and let VS_1 cut PQR in X.

Then, by Art. 2,

$$(P'Q'R'S') = (PQ_1R_1S_1) = (PQRX).$$

Hence, if $(P'Q'R'S') = (PQRS)$,

we have $(PQRX) = (PQRS)$.

$$\frac{PQ}{QR} \cdot \frac{RX}{XP} = \frac{PQ}{QR} \cdot \frac{RS}{SP}, \text{ i.e. } RX \,.\, PS = RS \,.\, PX,$$

i.e. $PS(RS - XS) = RS(PS - XS)$, i.e. $XS(PS - RS) = 0$,

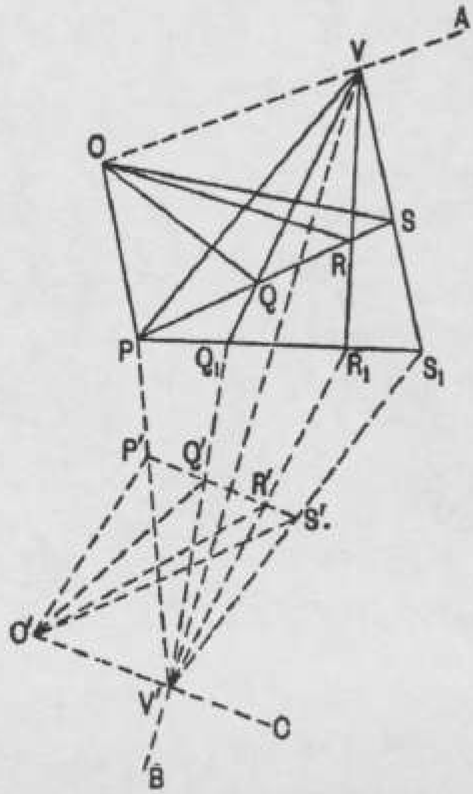
i.e. XS is zero, and hence X and S coincide.

The range $P'Q'R'S'$ therefore projects from the vertex V' into the range $PQ_1R_1S_1$, and then from the vertex V into the range $PQRS$.

200. *Two pencils can be projected into one another, provided their cross-ratios are equal.*

Let $PQRS$ and $P'Q'R'S'$ be any two transversals of the pencils, O and O' their vertices.

As in the last article, let $PQ_1R_1S_1$ be the range into which these can be projected by means of vertices V and V'.



From a vertex A, lying on OV, we can project the pencil $O(PQRS)$ into the pencil $V(PQ_1R_1S_1)$; from any vertex B, lying on VV', we can project the latter pencil into $V'(P'Q'R'S')$; and finally from any vertex C, lying on $O'V'$, we can project the latter pencil into $O'(P'Q'R'S')$.

201. We thus arrive at the very important proposition that any two ranges (or two pencils) can be projected into one another if their cross-ratios are the same, or, more shortly, **Homographic ranges, or pencils, are projective.**

Similarly pencils and ranges in involution are projective.

The double rays and double points project into the double rays and double points of the new involution pencils and ranges.

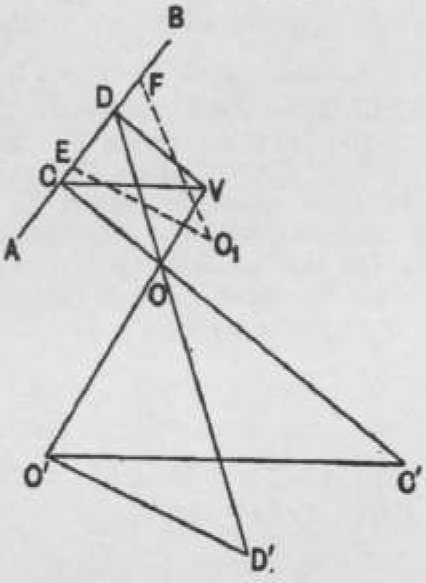
But the centre of an involution range does not project into the centre of the projected range.

202. As has been already stated in Art. 138, any set of four coplanar points (no three of which lie in a straight line) can be projected into any other four coplanar points in a different plane (no three of which lie in a straight line).

203. *To shew that any straight line of the original figure can be projected to infinity, and at the same time any two angles of this figure into two given angles, the plane of projection being any plane which is parallel to the given straight line, and the vertex of projection being properly chosen.*

Let AB be the straight line which is to be projected to infinity, and let OC, OD and O_1E, O_1F be the two pairs of straight lines which are to be projected into two pairs of straight lines containing given angles α and β respectively. Let C and D, E and F, be the intersections of these pairs of straight lines with AB.

Through AB draw a plane parallel to the plane of projection. In this plane so drawn describe segments of two circles, one through C and D containing an angle α, and the other through E and F containing an angle β.



Let one of the intersections, V, of these two segments be taken as the vertex of projection.

Let VO, CO and DO meet the plane of projection in the points O', C', and D'.

Then, since we have taken the plane VCD to be parallel to the plane of projection $O'C'D'$, and since any plane VCO is cut by parallel planes in parallel lines, therefore VC and $O'C'$ are parallel. Similarly VD and $O'D'$ are parallel. Hence $\angle C'O'D' = \angle CVD =$ the given angle α. Similarly, if the lines VO_1, EO_1 and FO_1 meet the plane in the points O_1', E' and F', the lines O_1E, O_1F will project into the lines $O_1'E'$, $O_1'F'$, and the angle between them will be equal to the angle EVF, *i.e.* β.

204. If the pairs of points C, D and E, F overlap, *i.e.* if one of the pair, C and D, lies between the pair E and F, the two circles to determine V always meet, and hence V is real and definite.

If neither of the pair, C and D, lies between the pair E and F, it may happen that the two circles do not meet, and then V is imaginary, and the projection is an imaginary one.

205. The foregoing proposition is of fundamental importance in the theory of projection. By its use we can project many figures with very much simpler forms.

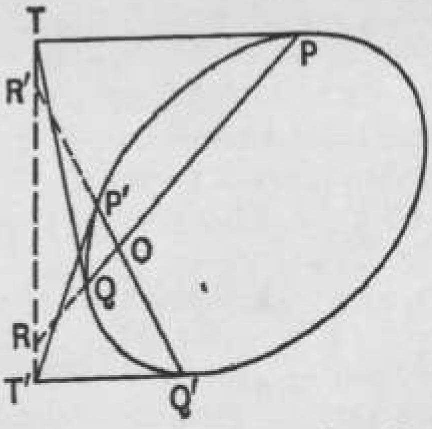
Ex. *To shew that a quadrilateral can be projected into a square.*

Let $PQRS$ be the given quadrilateral (Fig. Art. 15). Project the line BC joining two of its vertices, O_1 and O_2, to infinity, and at the same time the angles QPS and CAB into right angles. Since BC is projected to infinity, the sides PQ and RS are projected into parallel lines, as also the sides PS and QR. Hence the new figure is a parallelogram. But this parallelogram is a rectangle because one of its angles is a right angle; and it is then a square because the angle between its diagonals is a right angle.

206. *Any conic can be projected into a circle having the projection of any given point as centre.*

Let the given point be O, and through it draw two chords POQ, $P'OQ'$ of the given conic.

Let the tangents at P and Q meet in T, and the tangents at P' and Q' in T'.



Project the straight line TT' to infinity, and at the same time the angles TOR, $T'OR'$ into right angles, where PQ and $P'Q'$ meet TT' in R and R'.

Since O is on the polar of T, therefore T is on the polar of O. So T' is on the polar of O. Hence TT' is the polar of O.

Since the polar of O has been projected to infinity, O has been projected into the pole of the line at infinity, *i.e.* into the centre of the new conic.

Hence PQ projects into a diameter of the new conic, and OT into its conjugate diameter.

Similarly, $P'Q$ and OT' project into a pair of conjugate diameters.

Hence, in the projected conic, we have two pairs of conjugate diameters which are at right angles.

But no conic, except a circle, can have more than one pair of conjugate diameters at right angles.

Hence the projected conic is a circle.

If the point O is within the conic, the projection is always a real one.

Cor. 1. If the point O is projected into the centre of the circle, its polar, *i.e.* TT', with respect to the original conic is projected into the line at infinity.

Hence any conic can be projected into a circle, and at the same time any straight line can be projected to infinity, *i.e.* any straight line can be made to be the vanishing line.

This projection is always real if this vanishing line does not cut the original conic in real points.

Cor. 2. Any two points, U and V, can be projected into the circular points at infinity; for draw any conic through them, and project it into a circle and the straight line UV to infinity; the points U and V then project into the intersection of a circle and the line at infinity, *i.e.* into the circular points at infinity.

207. Ex. 1. *The cross-ratio of the range formed by four collinear points is the same as that of the pencil formed by their polars with respect to any conic.*

Project the conic into a circle; then the four points and their polars will project into four points and their polars with regard to the circle, and the cross-ratios of the range and the pencils are unaltered by projection.

But in a circle the line joining the centre O to any point P_1 is perpendicular to the polar of P_1, and so for the others. Hence the angle between any pair of lines OP_1, OP_2 is equal to that between their polars. Hence the cross-ratio of the pencil of polars is equal to that of the pencil OP_1, OP_2, OP_3, OP_4, and hence is equal to that of the range P_1, P_2, P_3, P_4.

Ex. 2. *The cross-ratio of the pencil formed by joining any variable point on a conic to four fixed points on the conic is constant, and equal to the cross-ratio of the range in which any variable tangent to the conic cuts the tangents at the four fixed points.*

As in the last example, on projection, we have only to prove the proposition for a circle. [See Art. 187.]

Circular points at infinity.

208. We have shewn (Part I, Art. 387) that all circles must be looked upon as passing through the same two points at infinity.

Also all conics through these two points are circles. For if $S=0$, $S'=0$ are two circles in Cartesian coordinates, the equation to any conic through their intersections is

$$S+\lambda S'=0.$$

Also if, in both S and S', the coefficients of x^2 and y^2 are equal and that of xy is zero, the same statements are true of $S+\lambda S'$, and hence $S+\lambda S'=0$ is a circle.

If then we call, for brevity, these circular points at infinity J and J', we see that

All circles pass through the same points at infinity J and J', and all conics through J and J' are circles.

209. If O be the origin of rectangular Cartesian coordinates, the equations to OJ and OJ' are $y=\pm\sqrt{-1}\,x$.

The cross-ratio of the range in which any two lines OA, OB meet $JJ'=e^{2\theta\sqrt{-1}}$, where $\angle AOB=2\theta$. (Art. 88, Ex. 6.)

Hence, if the two lines OA and OB are perpendicular, the ratio $=e^{\pi\sqrt{-1}}=\cos\pi+\sqrt{-1}\sin\pi=-1$, and the ratio is harmonic. Conversely, if this ratio is harmonic, we have $\cos 2\theta=-1$, and $\sin 2\theta=0$, and the lines are at right angles. Hence

Any two lines at right angles divide JJ' harmonically, and two lines which divide JJ' harmonically are at right angles.

It follows that J and J' are conjugate with respect to any rectangular hyperbola.

From the condition of Art. 39, it follows that the lines OJ, OJ' form two rays of the involution determined by any two pairs of lines, each pair of which is equally inclined to the same straight line.

210. Projective generalisations.

It is convenient to set down the relations of certain geometric facts and ideas with respect to the circular points at infinity and the line at infinity. For brevity, we shall speak of the circular points as J and J', and the line at infinity as JJ'.

Middle point R of a finite straight line PQ.	A point R which with the intersection of PQ with JJ' divides JJ' harmonically. [For $(PRQ\infty) = -1$.]
A point R which divides a given finite straight line PQ in a given ratio λ.	A point R which with the intersection of PQ with JJ' divides PQ in a given cross-ratio equal to $-\lambda$.
Parallel Lines.	Lines meeting on JJ'.
Angle of constant magnitude α.	Angle whose bounding lines divide JJ' in constant cross-ratio $e^{2\alpha i}$. (Art. 88, Ex. 6.)
Two lines at right angles.	Two lines which divide JJ' harmonically.
Bisectors of an angle POQ.	Lines through O which divide JJ' harmonically, and which also divide OP, OQ harmonically, *i.e.* they are the double lines of the involution pencil determined by OP, OQ and OJ, OJ'.
Centroid of a triangle.	Point of intersection of straight lines through the vertices of a triangle to points on the opposite sides, which with the intersections of the sides with JJ' divide the sides harmonically.
Ortho-centre of a triangle.	Point of intersection of the lines drawn from the vertices

	of a triangle which, with the opposite sides, divide JJ' harmonically.
A circle.	A conic through J and J'.
Centre of a circle.	Pole of JJ' with regard to a conic through J and J'.
Concentric circles.	Conics having double contact at the points J and J'.
Parabola.	Conic touching JJ'.
Rectangular Hyperbola.	Conic whose intersections with JJ' divide it harmonically.
Foci of a conic.	Intersections of the tangents to the conic from J and J'.
Directrix of a conic (*i.e.* Polar of the Focus).	Polar of the point of intersection of tangents from J and J'.
Principal axes of a conic.	Lines through the intersections of tangents to the conic from J and J', other than the tangents themselves.
Coaxal Circles.	Conics passing through J and J' and two other points A and B.
Circle on AB as diameter.	Conics passing through J, J', A, B, where AB and JJ' are conjugate lines.
Centre of similitude of two circles.	Intersection of a pair of common tangents to conics passing through J, J'.

211. Ex. 1. *The angles in the same segment of a circle are equal, and the angle in a semicircle is a right angle.*

Rewrite. The lines joining any point P on a conic through J, J' to fixed points Q and R on the conic together with the lines PJ, PJ' give a pencil of constant cross-ratio; and if the line QR passes through the pole of JJ' the pencil is harmonic.

Project the points J, J' into finite points F, F', and we have, *The lines joining any point P on a conic to four fixed points Q, F, R, F' on the conic give a pencil of constant cross-ratio; and, if QR and FF' are conjugate, the pencil is harmonic.*

Ex. 2. *The centres of a system of coaxal circles lie on a straight line perpendicular to the radical axis and bisecting it; also the orthogonal circles of the system have their centres on the radical axis.*

Rewrite. The locus of the poles of JJ', with regard to a system of conics passing through P, P', J, J', is a straight line passing through points U and V, where $(JUJ'O)$ and $(PVP'O)$ are harmonic, O being the intersection of PP' and JJ'; also the pole of JJ' with regard to any conic through J, J', the tangents at any one of whose other intersections with any conic of the system divide JJ' harmonically, lies on the line PP'.

Project J, J' into finite points F, F', *i.e.* substitute F, F' for J, J' in the preceding. It is easily seen that the points U, V lie on the third diagonal of the quadrangle.

Ex. 3. *If two triangles are circumscribed to a conic, prove that their vertices lie on another conic.*

Let the two triangles be ABC, $A'B'C'$. Through A, B, C, B', C' draw a conic and project it into a circle, and at the same time $B'C'$ into the line at infinity. Then B', C' become the circular points at infinity; the original conic becomes a parabola since it now touches the line at infinity; also A' reciprocates into its focus, since it is the point of intersection of tangents from the circular points.

Hence we have to prove that *If a triangle circumscribes a parabola its circum-circle passes through the focus of the parabola* (Art. 92, Ex. 3).

Ex. 4. *If two triangles are self-conjugate with regard to a conic, their six vertices lie on a conic, and their six sides touch a conic.*

Let the triangles be ABC and $A'B'C'$. Project the conic into a circle and at the same time BC into the line at infinity so that A, the pole of BC, becomes the centre of the circle. Also ABC will still be a self-conjugate triangle with respect to the circle, so that AB, AC will become perpendicular straight lines through the centre of the circle.

Also $A'B'C'$ will become a triangle self-conjugate with respect to the circle. But in a circle the line joining the centre of a circle is perpendicular to its polar. Hence AA', AB', AC' are perpendicular in the new figure to $B'C'$, $C'A'$, $A'B'$, so that A is now the ortho-centre of the new triangle $A'B'C'$.

A conic can be drawn through A', B', C', A and B and, since A is now the ortho-centre of $A'B'C'$, this conic is a rectangular hyperbola, which will then also pass through C since AB and AC are perpendicular.

Again a parabola can be drawn to touch the lines $B'C'$, $C'A'$, $A'B'$, AB and the line at infinity BC in the new figure. But the ortho-centre of a triangle whose sides touch a parabola is on its directrix. Hence A is on the directrix and, since AB is one tangent to the parabola, the perpendicular AC is also a tangent. Hence the six sides of the triangles touch a conic.

Ex. 5. *If a conic be inscribed in a quadrilateral the diagonals of this quadrilateral, and of the quadrilateral formed by the points of contact, meet in a point, and form a harmonic pencil.*

Project the conic into a circle, and the third diagonal to infinity, so that the quadrilateral becomes a parallelogram; the proposition at once follows, since all the diagonals now pass through the centre of the circle; also one pair of the diagonals are the bisectors of the angle between the other pair, and hence divide them harmonically.

Ex. 6. *By projection, prove Pascal's theorem that the three points of intersection of opposite sides of a hexagon inscribed in a conic are collinear.*

Project the conic into a circle and the line joining two of the points of intersection to infinity; we then have to prove that if the sides PQ, QR of a hexagon inscribed in a circle are parallel to the sides ST, TU, then the sides RS and UP are parallel. This follows at once, since

$$\begin{aligned}\angle SRU = \angle SRQ - \angle QRU &= \pi - \angle QTS - \angle RUT\\ &= \pi - \angle PQT - \angle RUT = \angle PUT - \angle RUT = \angle PUR.\end{aligned}$$

Ex. 7. If a conic passes through the vertices of a quadrangle, the locus of its centre is a conic through the middle points of the sides of the quadrangle.

If a conic passes through the vertices of a quadrangle, the locus of the pole of any given straight line is a conic passing through the fourth harmonics of the points in which the given line meets the sides of the quadrangle.

Ex. 8. If a quadrilateral can be inscribed in one conic S_1, and circumscribed to another conic S_2, an infinite number of conics can be so described.

An infinite number of quadrilaterals can be described about a conic with their angular points on its director circle.

[Project S_1 into a circle and the line joining two of the vertices of the quadrilateral to infinity; then the circle and the projection of S_2 have the same centre, viz. the third vertex; also any parallelogram inscribed in a circle is necessarily a rectangle, so that the circle is the director circle of the new conic. Since the projected theorem is true, it follows that the original theorem is also true.]

Ex. 9. The locus of the foot of the perpendicular upon any tangent to a central conic from a focus is a circle.

The tangents from two fixed points F, F' to a conic meet in S; the locus of the intersection of any tangent, P, to the conic with a line through S, which with P divides FF' harmonically, is a conic passing through F and F'.

EXAMPLES.

Generalise the following twelve theorems by projection:

1. The tangent at any point of a circle is perpendicular to the radius to the point of contact.

2. Any diameter of a circle is bisected at its centre.

3. If a straight line touches a circle, and from the point of contact a straight line is drawn cutting the circle, the angles which this line makes with the tangent are equal to the angles in the alternate segments of the circle.

4. Chords of a circle which subtend equal angles at (1) the centre, (2) the circumference, envelope a concentric circle.

5. If two circles are concentric, any chord of one which touches the other is bisected at the point of contact.

6. The feet of the perpendiculars drawn to the sides of a triangle inscribed in a circle from any point on the circle are collinear.

7. If PSP' be a focal chord of a conic and the tangents at P, P' meet in Z, then Z lies on the directrix and ZSP is a right angle.

8. The envelope of a chord of a conic which subtends a constant angle at the focus, and the locus of its pole, are conics with the same focus and directrix as the original conic.

9. The locus of the intersection of two tangents to a parabola, which meet at a given angle, is a hyperbola with the same focus and directrix.

10. If a focus and two tangents to a conic are given, the locus of the other focus is a straight line.

11. Given the focus and two points on a conic, the directrix passes through one or other of two fixed points.

12. Given three concentric circles; any tangent to one of them is cut by the other two in a range of constant cross-ratio.

13. A triangle ABC can be so projected that any three lines through its vertices, concurrent in a point O, shall become the medians of the triangle in the projection.

[Project the polar line of O to infinity, Art. 60, Exs. 5, 6.]

14. Any triangle can be projected into an equilateral triangle by a real projection.

15. Any conic can be so projected that two given points S and S' shall project into a pair of foci.

[Draw tangents from S and S' to meet in T and T'; through T and T' draw any conic; project this conic into a circle, and the line TT' to infinity.]

16. Any three chords AA', BB', and CC' of a conic can be projected into equal chords of a circle.

[Project the conic into a circle and the line joining the intersections of AB', $A'B$ and of BC', $B'C$ to infinity.]

If A' coincide with B, B' with C, and C' with A, deduce that a conic and an inscribed triangle can always be projected into a circle and an inscribed equilateral triangle.

17. Conics having a common focus and directrix can be projected into concentric circles. [For they are conics having double contact.]

18. Any two conics can be projected into circles. [Project one conic into a circle, and the common chord of the two conics to infinity.]

19. Any two conics can be projected into concentric conics. [Project a side BC of the common self-conjugate triangle ABC to infinity. If, in addition, BAC is projected into a right angle, we have concentric coaxal conics. If B, C are projected into the circular points at infinity, we have concentric rectangular hyperbolas.]

20. Rectangular hyperbolas can be projected into conics having two given points F, F' as conjugate points. [For the intersections of a rectangular hyperbola with JJ' divide JJ' harmonically.]

212. *A system of conics inscribed in a quadrilateral can be projected into confocal conics.*

Let the quadrilateral be $PQRS$, as in the figure of Art. 15, and let two of the sides intersect in O_1, and the other two in O_2.

Draw any conic through the points O_1 and O_2, and project it into a circle, and at the same time the line O_1O_2 into the line at infinity.

Then O_1 and O_2 are projected into the circular points at infinity, and the conics into a set of conics inscribed in the projected quadrilateral. Since the sides of the new quadrilateral pass through the circular points at infinity, their other points of intersection are the four foci, two real and two imaginary, of any conic inscribed in the new quadrilateral.

Since the foci of the new system of conics are the same, the conics form a confocal system.

Hence the proposition.

Cor. Any two conics can be projected into confocal conics. For they are inscribed in the quadrilateral formed by their four common tangents.

213. *A system of conics which pass through four given points can be projected into a system of coaxal circles.*

Let the four given points be P, Q, R, S. Project the line joining any two of them, say P and R, to infinity, and one of the conics into a circle. Then P and R will project into the circular points at infinity, and through their projections will pass all the projected conics; these projected conics will therefore be all circles, since they pass through the circular points at infinity. Also these circles will all pass through the projections of the other points Q and S, and will thus be a coaxal system.

Cor. Conics which have double contact with another can be projected into concentric circles.

For, in this case, the conics will pass through two coincident points at P, and two coincident points at R. Hence they will project into circles touching one another at the circular points at infinity, *i.e.* they will become concentric circles (Part I, Art. 388).

214. Ex. 1. *A quadrangle can be so projected that its sides and two of its diagonals become the three sides and the three perpendiculars of a triangle, and hence all conics through four points can be projected into rectangular hyperbolas.*

Let the quadrangle be as in the figure of Art. 64. Project the angles PBQ, QCR into right angles. Then in the new figure we have four points, such that each is the ortho-centre of the triangle formed by the other three. Also the lines joining the four points, viz. the sides PQ, QR, RS, and SP and its two diagonals QS and PR, become the sides and perpendiculars in the new figure.

The conics of the original figure become in the projection conics circumscribing a triangle, and passing through its ortho-centre, and hence the latter are all rectangular hyperbolas.

Ex. 2. *If a system of conics pass through four points they will cut any transversal in involution.* (Desargues' Theorem.)

Project the conics into coaxal circles. Then, if any transversal cuts the common radical axis in O, any one of the circles in P and P', and another in Q and Q', we know that the rectangles $OP \, . \, OP'$ and $OQ \, . \, OQ'$ are equal; for each is equal to the square of the tangent from O to all the circles. Hence, (Art. 29), we have an involution.

Ex. 3. *If two conics be inscribed in a quadrilateral, two of whose vertices are F and F', the tangents at their common points divide FF' harmonically.*

Project F and F' into the circular points at infinity, so that the conics become confocal conics; and we have to prove that the tangents at a common point cut JJ' harmonically, *i.e.* we have to prove that *Confocal conics cut at right angles.*

Ex. 4. The locus of the pole of a straight line P, with regard to a system of confocals, is the normal to the confocal, which P touches, at the point of contact of P.	The locus of the pole of a straight line P, with regard to a system of conics, touching the sides of a quadrilateral $SFS'F'$, is a straight line passing through the point of contact of P with one of the conics, and which with P divides FF' harmonically.
Ex. 5. The locus of the intersection of two perpendicular tangents to a conic is a concentric circle; and the envelope of the chord of contact is a confocal conic.	The locus of the point of intersection of two tangents to a conic, which divide the line joining two given points F, F' harmonically, is a conic passing through F, F', the pole of FF' with regard to these two conics being the same; also the envelope of the chord of contact is another conic touching the tangents from F and F' to the original conic.
Ex. 6. If the original conic is a parabola, the locus is the directrix and the envelope reduces to the focus.	If the original conic touches FF', the locus is the line joining the points of contact of the other tangents to it from F and F'; and the chord of contact passes through the point of intersection of these tangents from F, F'.

EXAMPLES.

By projection prove the following six theorems:

1. If the lines joining the vertices of two triangles are concurrent, then the corresponding sides meet in three points which are collinear.

2. Two conics have double contact at F and F', and C is the pole of FF'; a tangent to one conic at P meets the other in A and B and FF' in Q; prove that $C(APBQ)$ is harmonic.

3. Two sides of a triangle inscribed in a conic pass through fixed points P and Q; shew that the envelope of the third side is a conic having double contact with the given one. [Project the conic into a circle and the line PQ into the line at infinity.]

4. If three conics pass through the same four points, the common tangent to any two is cut harmonically by the third conic.

5. If two conics have double contact, they have an infinite number of self-conjugate triangles. [Project the line joining the points of contact to infinity and we have two concentric circles; and any two perpendicular lines through the centre together with the line at infinity form a self-conjugate triangle for the concentric circles.]

6. The diagonal triangle of a quadrilateral which circumscribes a conic is self-conjugate with respect to the conic. [Project the quadrilateral into a square.]

Generalise the following eight theorems by projection:

7. If two circles meet in two real points, the common chord bisects the common tangents.

Deduce that, if PQ is the common chord of an ellipse and its circle of curvature at P, then PQ, the tangent at P and the lines joining P to the points of contact of a common tangent form a harmonic pencil.

8. Chords of a conic, which subtend a right angle at a given point P of the conic, pass through a fixed point on the normal at P. (Frégier Point.)

9. The locus of the centre of a rectangular hyperbola which circumscribes a triangle is the nine-point circle of the triangle; also, if the hyperbola is self-conjugate with respect to the triangle, the locus is the circum-circle.

10. The directrix of a parabola which touches the sides of a given triangle passes through its ortho-centre; also, if the parabola is self-conjugate to the triangle, the directrix passes through the circum-centre.

11. The locus of the foot of the perpendicular let fall from the focus upon the normal at any point P of a parabola is another parabola coaxal with the former.

12. A point P moves on a straight line; the locus of the centre of the circle which passes through the feet of the three normals drawn from P to a parabola is also a straight line.

13. The circum-circle of a triangle which is self-conjugate with regard to a conic cuts its director-circle orthogonally.

Also reciprocate the theorem with respect to a point on the director-circle.

14. The envelope of the polar of a given point for a system of confocals is a parabola, touching the axes of the confocals, and having the given point on its directrix.

15. Generalise, both by projection and reciprocation, the theorem that, if two circles cut orthogonally, the ends of any diameter of either circle are conjugate with respect to the other circle.

CHAPTER VIII

INVARIANTS

215. If

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

and $S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0$

are the equations to any two conics, the equation to any conic through their points of intersection is

$$kS + S' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The condition that this latter conic should be a pair of straight lines is

$$(ka + a')(kb + b')(kc + c') + 2(kf + f')(kg + g')(kh + h')$$
$$-(ka + a')(kf + f')^2 - (kb + b')(kg + g')^2 - (kc + c')(kh + h')^2 = 0,$$

i.e. $\Delta k^3 + \Theta k^2 + \Theta' k + \Delta' = 0 \quad \ldots\ldots\ldots\ldots(2),$

where

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = Aa + Hh + Gg;$$

$$\Theta = a'(bc - f^2) + b'(ca - g^2) + c'(ab - h^2)$$
$$+ 2f'(gh - af) + 2g'(hf - bg) + 2h'(fg - ch)$$
$$= Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh';$$

$$\Theta' = a(b'c' - f'^2) + b(c'a' - g'^2) + c(a'b' - h'^2)$$
$$+ 2f(g'h' - a'f') + 2g(h'f' - b'g') + 2h(f'g' - c'h')$$
$$= A'a + B'b + C'c + 2F'f + 2G'g + 2H'h;$$

and

$$\Delta' = a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2 = A'a' + H'h' + G'g'.$$

Here Δ, Δ' are the discriminants of the two conics, and the capital letters have the usual meanings as defined in Art. 72.

The equation for k has therefore, in general, three roots corresponding to the three pairs of straight lines that can be drawn through the four (real or imaginary) intersections of the conics.

If we eliminate k between (1) and (2), we have, as the equation to these three pairs of straight lines,

$$\Delta S'^3 - \Theta S'^2 S + \Theta' S' S^2 - \Delta' S^3 = 0.$$

The coordinates in which S and S' are expressed may be of any kind—Trilinear or Areal or Cartesian. If the latter coordinates are used, we merely put z equal to unity.

216. If on transferring to any new system of coordinates, of any kind whatsoever, the quantities S and S' become S_1 and S_1', then clearly $kS + S'$ will become $kS_1 + S_1'$, and k is unaltered. If, in addition, $kS + S'$ represented a pair of straight lines, it must still continue to do so. Hence the equation giving the values of k, for which $kS + S'$ represents straight lines, will be the same as that giving the values of k, for which $kS_1 + S_1'$ represents straight lines. Hence the roots of equation (2) of the previous article must remain unaltered by the transformation, and so the ratios of any two of the four quantities Δ, Θ, Θ', and Δ' must remain unaltered, and be independent of the particular axes or system of coordinates.

For these reasons these quantities Δ, Θ, Θ', and Δ' are called the **Invariants** of the two conics $S = 0$ and $S' = 0$.

In the case of any two conics if we have calculated the values of these four quantities, and find any homogeneous relation to hold between them, we can be certain that the same relation will exist between them however we may transform the axes of coordinates.

217. In the course of the transformation it may however happen that one equation has been multiplied by a constant quantity, but not the other. Thus, suppose that the coefficients of the terms in $S' = 0$ had been multiplied by a quantity p, and that the terms in $S = 0$ had not been multiplied by p, then, since Θ, Θ', and Δ' contain the coefficients of S' in the first, second, and third degrees respectively, the new quantities Θ, Θ', Δ' would have been multiplied by p, p^2 and p^3 respectively. Any homogeneous relation between the original invariants would not necessarily remain true. Thus the homogeneous relation $\Theta\Delta = \Theta'\Delta'$ would become

$p\Theta_1 \times \Delta_1 = p^2\Theta_1' \times p^3\Delta_1'$, which is not of the same form as before But the homogeneous relation $\Theta\Theta' = \Delta\Delta'$ would become $p\Theta_1 \times p^2\Theta_1' = \Delta_1 \times p^3\Delta_1'$, *i.e.* $\Theta_1\Theta_1' = \Delta_1\Delta_1'$, which is of the same form as before.

Hence, that we may be sure that a relation between the above four quantities remains the same whatever the transformation, including the case when S and S' are multiplied by different constant quantities, the relation must be homogeneous both when Δ, Θ, Θ', and Δ' are considered as of the same dimensions, and also when they are assumed to be of dimensions 0, 1, 2, and 3 respectively, or when they are assumed to be of dimensions 3, 2, 1, and 0 respectively.

218. For convenience of reference the values of the invariant quantities are written down for certain standard cases of frequent occurrence.

In-conic and circum-conic.

$$S \equiv l^2x^2 + m^2y^2 + n^2z^2 - 2mnyz - 2nlzx - 2lmxy;$$
$$S' \equiv 2f'yz + 2g'zx + 2h'xy.$$
$$\Delta = -4l^2m^2n^2; \ \Theta = 4lmn\,(lf' + mg' + nh');$$
$$\Theta' = -(lf' + mg' + nh')^2; \text{ and } \Delta' = 2f'g'h'.$$

In-conic and self-conjugate conic.

$$S \equiv l^2x^2 + m^2y^2 + n^2z^2 - 2mnyz - 2nlzx - 2lmxy;$$
$$S' \equiv Lx^2 + My^2 + Nz^2.$$
$$\Delta = -4l^2m^2n^2; \ \Theta = 0;$$
$$\Theta' = l^2MN + m^2NL + n^2LM; \text{ and } \Delta' = LMN.$$

Circum-conic and self-conjugate conic.

$$S \equiv 2fyz + 2gzx + 2hxy;$$
$$S' \equiv Lx^2 + My^2 + Nz^2.$$
$$\Delta = 2fgh; \ \Theta = -(Lf^2 + Mg^2 + Nh^2); \ \Theta' = 0; \ \Delta' = LMN.$$

Two self-conjugate conics.

$$S \equiv Lx^2 + My^2 + Nz^2;$$
$$S' \equiv L'x^2 + M'y^2 + N'z^2.$$
$$\Delta = LMN; \ \Theta = L'MN + M'NL + N'LM;$$
$$\Theta' = LM'N' + MN'L' + NL'M'; \ \Delta' = L'M'N'.$$

Ellipse and Circle.

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1;$$

$$S' \equiv (x-\alpha)^2 + (y-\beta)^2 - r^2.$$

$$\Delta = -\frac{1}{a^2b^2}; \quad \Theta = \frac{\alpha^2 + \beta^2 - a^2 - b^2 - r^2}{a^2b^2};$$

$$\Theta' = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right); \ \Delta' = -r^2.$$

Two circles.

$$S \equiv (x-\alpha)^2 + (y-\beta)^2 - r^2.$$

$$S' \equiv (x-\alpha')^2 + (y-\beta')^2 - r'^2.$$

$$\Delta = -r^2; \ \Theta = (\alpha-\alpha')^2 + (\beta-\beta')^2 - 2r^2 - r'^2;$$

$$\Theta' = (\alpha-\alpha')^2 + (\beta-\beta')^2 - r^2 - 2r'^2; \ \Delta' = -r'^2.$$

Parabola and Circle.

$$S \equiv y^2 - 4px;$$

$$S' \equiv (x-\alpha)^2 + (y-\beta)^2 - r^2.$$

$$\Delta = -4p^2; \ \Theta = -4p(p+\alpha); \ \Theta' = \beta^2 - 4p\alpha - r^2; \ \Delta' = -r^2.$$

219. We shall now examine the geometrical meaning of the vanishing of some of the invariants, and of certain relations between them.

$\Theta = 0$.

From the value written down in Art. 215, it is clear that Θ vanishes when $f = g = h = 0$ and at the same time $a' = b' = c' = 0$.

S then becomes $ax^2 + by^2 + cz^2 = 0$, *i.e.* a conic self-conjugate to the triangle of reference, and S' becomes

$$2f'yz + 2g'zx + 2h'xy = 0,$$

i.e. a conic circumscribed to this triangle.

Hence Θ *vanishes when a triangle self-conjugate to S can be inscribed in S'.*

From the same article it is clear that Θ also vanishes when $bc = f^2$, $ca = g^2$, $ab = h^2$ and at the same time $f' = g' = h' = 0$, so that S is an inscribed conic and S' is a self-conjugate conic.

Hence Θ *also vanishes when a triangle can be drawn to circumscribe S and also to be self-conjugate to S.*

220. Conversely, *If* $\Theta=0$, *then* (1) *an infinite number of triangles, self-conjugate to* S, *can be inscribed in* S', *and*

(2) *an infinite number of triangles can be drawn to circumscribe* S *and be self-conjugate to* S'.

To prove (1), take any point A on S', and let its polar with regard to S meet S' in B and C. With ABC as triangle of reference, we have

$$S \equiv ax^2+by^2+cz^2+2fyz=0,$$

and $\quad S' \equiv 2f'yz+2g'zx+2h'xy=0.$

[For g and h are both zero, since the polar of A is $x=0$.]

Since $\quad \Theta=0, \therefore -2aff'=0.$

But f' cannot be zero; for then S' would be two straight lines; neither can a be zero, for a similar reason.

Hence $f=0$, and thus the triangle of reference, which is inscribed in S', is also self-conjugate with respect to S.

Since A is any point on S', there are an infinite number of such triangles.

To prove (2), let BC be any tangent to S, whose pole with regard to S' is A, and let AB and AC be tangents to S. With ABC as triangle of reference, we have

$$S \equiv l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy=0,$$

and $\quad S' \equiv a'x^2+b'y^2+c'z^2+2f'yz=0.$

Since $\quad \Theta=0, \therefore 4l^2mnf'=0.$

But neither l, m, nor n can be zero; for then S would be a pair of coincident straight lines.

Hence $f'=0$, and thus the triangle of reference, which is circumscribed to S, is also self-conjugate to S'.

Since BC is any tangent to S', there are an infinite number of such triangles.

221. $\Theta'=0$.

From the values in Art. 215, it appears that Θ' vanishes when $f=g=h=0$ and at the same time $b'c'=f'^2$, $c'a'=g'^2$ and $a'b'=h'^2$, so that S is a self-conjugate conic and S' an inscribed conic.

Hence Θ' *vanishes when a triangle self-conjugate to* S *can be circumscribed about* S'.

Again, Θ' also vanishes when $a=b=c=0$, and at the same time $f'=g'=h'=0$, *i.e.* when S is a circumscribing conic of the triangle of reference, and S' a self-conjugate conic.

Hence Θ' *also vanishes when a triangle can be inscribed in* S *which is self-conjugate to* S'.

As in the last article, the converses of the above theorems can be shewn to be true.

Ex. Let S be the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and S' the parabola

$$\sqrt{\frac{x}{c}}+\sqrt{\frac{y}{d}}=1.$$

222. $\Theta^2=4\Delta\Theta'$.

Suppose we want the condition that a triangle can be circumscribed to one conic S and inscribed in another conic S'. Assuming that such a triangle exists, take it for the triangle of reference.

Then

$$S\equiv l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy=0,$$

and

$$S'\equiv 2f'yz+2g'zx+2h'xy=0.$$

Then, from the results in Art. 218, it is clear that

$$\Theta^2=4\Delta\Theta'.$$

This relation is homogeneous in Δ, Θ, Θ' and Δ'; it is also homogeneous if they are assumed to be of dimensions 0, 1, 2, 3 respectively; for then each side is of dimensions 2.

It therefore is an invariantal relation of the form spoken of in Art. 217, and hence this relation must always exist between the equations of two conics such as S and S'. Hence

If a triangle can be circumscribed to $S=0$, and also be inscribed in $S'=0$, then $\Theta^2=4\Delta\Theta'$.

Conversely, *If $\Theta^2=4\Delta\Theta'$, then a triangle, and any number of triangles can be circumscribed to $S=0$ which are inscribed in $S'=0$.*

For let any tangent to S meet S' in the points B, C, and let the other tangents to S from B and C meet in A. Then A shall also lie on S'. Take ABC as the triangle of reference.

Then $\quad S\equiv l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy,$

and $\quad S'\equiv a'x^2+2f'yz+2g'zx+2h'xy.$

[For S is an inscribed conic, and S' goes through B and C.]

Hence $\quad \Delta=-4l^2m^2n^2;\quad \Theta=4lmn(lf'+mg'+nh');$

$$\Theta'=-(lf'+mg'+nh')^2+2mna'f';\quad \Delta'=f'[2g'h'-a'f'].$$

Hence, since $\Theta^2=4\Delta\Theta'$, we have $l^2m^2n^2a'f'=0$.

Neither l, m nor n can be zero; for then S would only represent a pair of coincident straight lines; nor can f' be zero; for then S' would be a pair of straight lines. Hence we must have $a'=0$, and so $S'=0$ also passes through A.

Since BC is any tangent to S', an infinite number of such triangles can be drawn.

223. $\Theta'^2=4\Delta'\Theta$.

This similarly is the condition that a triangle can be inscribed in $S=0$, and circumscribed to S'.

224. $\Theta=0$ and also $\Theta'=0$.

When both these relations are satisfied, then the four preceding cases are all included, and we have that

An infinite number of triangles can be inscribed in $\left.\begin{matrix}S=0\\S'=0\end{matrix}\right\}$ *which are circumscribed to* $\left.\begin{matrix}S'=0\\S=0\end{matrix}\right\}$; *and an infinite number of triangles can be inscribed in or circumscribed to* $\left.\begin{matrix}S=0\\S'=0\end{matrix}\right\}$ *which are self-conjugate for* $\left.\begin{matrix}S'=0\\S=0\end{matrix}\right\}$.

Numerical Example. Let

$$S\equiv\beta\gamma-k_1\alpha^2=0 \text{ and } S'\equiv\gamma\alpha-k_2\beta^2=0.$$

225. *If two triangles are self-conjugate with respect to a conic $S=0$, a conic can be described to pass through the six vertices of the triangles, and also another conic to touch their six sides.*

Let ABC and $A'B'C'$ be the two conics, and take ABC as the triangle of reference.

Then $$S\equiv ax^2+by^2+cz^2=0.$$

Let a second conic S' pass through the five points A', B', C', B and C, so that

$$S'\equiv a'x^2+2f'yz+2g'zx+2h'xy.$$

Since $A'B'C'$ is a triangle self-conjugate to S and inscribed in S',

$$\therefore\ \Theta=0,\ \textit{i.e.}\ a'bc=0.$$

$\therefore\ a'=0$, and the conic S' passes through A also.

Again, let a conic S'' touch the three sides of the triangle $A'B'C'$, and also the sides AB, AC.

Then

$$S''\equiv a''x^2+b''y^2+c''z^2+2f''yz\pm2\sqrt{c''a''}zx\pm2\sqrt{a''b''}xy=0.$$

Since $A'B'C'$ is a triangle self-conjugate to S and circumscribed to S'', therefore the Θ' of S and S'' is zero, *i.e.*

$$a(b''c''-f''^2)=0; \quad \therefore b''c''=f''^2.$$

$\therefore S''$ also touches BC.

226. Gaskin's Theorem. *The circum-circle of a triangle which is self-conjugate with regard to a conic cuts the director circle of the conic orthogonally.*

Let $$S\equiv ax^2+by^2-1=0,$$
and $$S'\equiv(x-\alpha)^2+(y-\beta)^2-r^2=0.$$

If a triangle, self-conjugate to S, can be inscribed in S', then $\Theta=0$.

Hence, as in Art. 218,

$$\alpha^2+\beta^2-r^2-\frac{1}{a}-\frac{1}{b}=0 \text{ ………… …………(1).}$$

$\therefore$ the square of the distance between the centres of S and S'

$$=\alpha^2+\beta^2=r^2+\frac{1}{a}+\frac{1}{b}$$

$=r^2+$ the square of the radius of the director circle of S.

Hence S', the circum-circle of the triangle, and the director circle of the conic cut orthogonally.

Again, from Art. 219, we know that $\Theta=0$ if a triangle can be circumscribed to $S=0$ which is self-conjugate to $S'=0$. Hence (1) also shews that *the self-conjugate circle of any triangle which is circumscribed to a conic cuts its director circle orthogonally.*

Particular Cases. I. Let the conic in each case be a parabola. Then we have

(1) The centre of the circle circumscribing a triangle, which is self-conjugate with regard to a parabola, lies on its directrix, and

(2) The centre of the circle which is self-conjugate with respect to a triangle, which circumscribes a parabola, lies on its directrix, *i.e.* the ortho-centre of a triangle which circumscribes a parabola lies on its directrix.

II. Let the conic in each case be a rectangular hyperbola. Then we have

(1) The circum-circle of a triangle, self-conjugate with respect to a rectangular hyperbola, passes through its centre, and

(2) The self-conjugate circle of a triangle, which circumscribes a rectangular hyperbola, passes through its centre.

III. The circle self-conjugate with regard to a given triangle cuts orthogonally the circles on the sides as diameters.

[For the sides are particular cases of inscribed conics.]

IV. The circles on the diagonals of a complete quadrilateral as diameters are cut orthogonally by the circum-circle of the triangle formed by the diagonals of the quadrilateral.

EXAMPLES.

1. If a triangle be self-conjugate with respect to a parabola, its circum-centre lies on the directrix.

[Let $S\equiv y^2-4ax=0$, and $S'\equiv(x-\alpha)^2+(y-\beta)^2-r^2=0$.
If a triangle self-conjugate with respect to S is inscribed in S', then $\Theta=0$, which gives $\alpha=-a$, etc.]

2. If a triangle be circumscribed to a parabola, the centre of its self-conjugate circle lies on the directrix. [Here again $\Theta=0$.]

3. If a triangle be self-conjugate with respect to a rectangular hyperbola, the latter passes through the in-centre of the triangle.

[Take $S\equiv(x-\alpha)^2+(y-\beta)^2-r^2=0$; $S'\equiv 2xy-k^2=0$; then $\Theta=0$.]

4. If a triangle be circumscribed to a rectangular hyperbola, shew that the circle with regard to which the triangle is self-conjugate passes through the centre of the hyperbola.

[Take $S\equiv 2xy-k^2=0$; $S'\equiv x^2+y^2-2gx-2fy+c=0$.
Then we are given that $\Theta=0$, and hence that $c=0$.]

5. If $F=0$ [Art. 113] be the harmonic locus of two conics $S=0$ and $S'=0$, and if any triangle inscribed in S' is self-conjugate with respect to S, then any triangle inscribed in F is self-conjugate with respect to S'.

6. What is the meaning of the relation $\Theta=0$, according as either one or each of the quantities Δ, Δ' is zero?

What do we know of the conics $S=0$, $S'=0$ if Δ, Θ, Θ' and Δ' are all zero?

7. If the directrix of a parabola passes through the centre of a circle, shew that an infinite number of triangles can be drawn to circumscribe the parabola and be self-conjugate with respect to the circle; and also an infinite number of triangles can be inscribed in the circle which are self-conjugate with regard to the parabola.

8. Shew that an infinite number of triangles can be circumscribed to the conic $S=0$, and inscribed in the conic $S'=0$, in the following cases:

(1) $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$; $S'\equiv x^2+y^2-(a+b)^2=0$;

(2) $S\equiv x^2+y^2-\frac{a^2b^2}{(a+b)^2}=0$; $S'\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$;

(3) $S\equiv y^2-4ax$; $S'\equiv y^2-ax-by+c=0$;

(4) when S is a parabola, and S' any circle passing through its focus.

9. Shew that the locus of the centre of the self-conjugate circle of a triangle circumscribed to a given conic is a circle if the radius of the self-conjugate circle is given.

10. If the director circle of a conic inscribed in a triangle touches the circum-circle of the triangle it also touches the nine-point circle.

[Let O, N, P be the circum-centre, nine-point centre, and ortho-centre of a triangle, and K the centre of the conic. Let t and ρ be the radii of the director circle of the conic and of the self-conjugate circle of the triangle. Then, by the second theorem of Art. 226, $KP^2=t^2+\rho^2$.

If the director circle touches the circum-circle, then $OK=t+R$.

$$\begin{aligned}\therefore (t^2+\rho^2)+(t+R)^2=KP^2+OK^2&=2NK^2+\tfrac{1}{2}OP^2\\&=2NK^2+\tfrac{1}{2}(R^2-8R^2\cos A\cos B\cos C)\\&=2NK^2+\tfrac{1}{2}(R^2+2\rho^2).\end{aligned}$$

$$\therefore NK=t+\frac{R}{2},$$

i.e. the director circle and nine-point circle also touch.]

227. *To shew that two conics, $S=0$ and $S'=0$, will touch if*

$$(\Theta\Theta'-9\Delta\Delta')^2=4(\Theta^2-3\Delta\Theta')(\Theta'^2-3\Delta'\Theta);$$

and that they will have contact of the second order if

$$\Theta^2=3\Delta\Theta' \text{ and } \Theta'^2=3\Delta'\Theta.$$

If two conics touch then, as in Part I, Art. 385, two of the pairs of straight lines through their points of intersection coincide. Hence, instead of three different pairs of straight lines through their points of intersection, we only have two, and therefore the equation in k of Art. 215 has two roots equal. Let the roots be a, a and β. Then

$$2a+\beta=-\frac{\Theta}{\Delta};\ 2a\beta+a^2=\frac{\Theta'}{\Delta};\text{ and }a^2\beta=-\frac{\Delta'}{\Delta}.$$

$$\therefore\ 2a.\frac{\Theta}{\Delta}+\frac{\Theta'}{\Delta}=-3a^2,\text{ and }\frac{\Theta}{\Delta}a^2+\frac{2\Theta'}{\Delta}a=3a^2\beta=-\frac{3\Delta'}{\Delta}.$$

$$\therefore\ 3a^2\Delta+2a\Theta+\Theta'=0\ \ldots\ldots\ldots\ldots\ldots(1),$$

and

$$a^2\Theta+2a\Theta'+3\Delta'=0\ \ldots\ldots\ldots\ldots\ldots(2).$$

Solving (1) and (2), we have

$$\frac{a^2}{6\Delta'\Theta-2\Theta'^2}=\frac{a}{\Theta\Theta'-9\Delta\Delta'}=\frac{1}{6\Delta\Theta'-2\Theta^2}.$$

$$\therefore\ [\Theta\Theta'-9\Delta\Delta']^2=4[\Theta^2-3\Delta\Theta'][\Theta'^2-3\Delta'\Theta].$$

[Equation (1) also follows from the fact that if any equation $\phi(k)=0$ has equal roots, each of them is a root of $\frac{d\phi}{dk}=0$.]

If the conics have contact of the second order, there is similarly only one pair of straight lines through their points of intersection. In this case the roots are all equal to a, and we have

$$3a=-\frac{\Theta}{\Delta};\ 3a^2=\frac{\Theta'}{\Delta};\ \text{and}\ a^3=-\frac{\Delta'}{\Delta}.$$

$$\therefore\ \Theta^2=3\Delta\Theta',\ \text{and}\ \Theta'^2=3\Delta'\Theta.$$

These relations are of the right dimensions according to the rule of Art. 217.

228. *To find the condition that a quadrilateral can be drawn to circumscribe a conic $S=0$ and be inscribed in a conic $S'=0$.*

We know that a quadrilateral can always be projected into a rectangle; hence, after projection, S is a conic and S' is its orthogonal circle.

Thus
$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

and
$$S'\equiv x^2+y^2-a^2-b^2=0.$$

Hence, as in Art. 218, we have

$$\Delta=-\frac{1}{a^2b^2};\ \Theta=-\frac{2a^2+2b^2}{a^2b^2};\ \Theta'=-1-\frac{(a^2+b^2)^2}{a^2b^2};$$

and
$$\Delta'=-(a^2+b^2).$$

$$\therefore\ \Theta^2-4\Delta\Theta'=-\frac{4}{a^2b^2}=-8\frac{\Delta^2\Delta'}{\Theta}.$$

$$\therefore\ \Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta'=0.$$

This relation is of the proper dimensions according to the rule of Art. 217.

If one rectangle can be drawn circumscribing S its vertices lie on the director circle, and hence, from the general property of the director circle, it follows that an infinite number of such rectangles can be drawn.

Cor. Two conics are such that quadrilaterals can be circumscribed to either, with the ends of two diagonals on the other, if

$$\Theta^2 = 2\Delta\Theta' \text{ and } \Theta'^2 = 2\Delta'\Theta.$$

For, in this case, we have both

$$\Theta^3 - 4\Delta\Theta\Theta' + 8\Delta^2\Delta' = 0,$$

and $$\Theta'^3 - 4\Delta'\Theta\Theta' + 8\Delta\Delta'^2 = 0,$$

and these are both satisfied when

$$\Theta^2 = 2\Delta\Theta', \text{ and } \Theta'^2 = 2\Delta'\Theta.$$

229. Exs. Prove that the relation

$$\Theta^3 - 4\Delta\Theta\Theta' + 8\Delta^2\Delta' = 0$$

is also satisfied,

(1) when the tangents to S at two of its points of intersection with S' intersect on S';

(2) when S is inscribed in the triangle formed by two tangents to S' and their chord of contact;

(3) when S' is the locus of the intersection of tangents to S which divide a given line FF' harmonically;

(4) when the other tangents drawn to S from the points where any tangent to it cuts S' intersect on a given straight line.

230. *Two conics S and S' are such that, if two of their points of intersection are joined to either of the two others, the two chords and the two tangents at that point form a harmonic pencil; prove that $\Theta\Theta' = \Delta\Delta'$. Deduce the condition that two circles cut orthogonally.*

Take three of the points of intersection A, B, C as the triangle of reference, and let the conics be

$$S \equiv 2fyz + 2gzx + 2hxy = 0,$$

and $$S' \equiv 2f'yz + 2g'zx + 2h'xy = 0.$$

The tangents to the conics at the point A are $gz + hy = 0$ and $g'z + h'y = 0$, and these with AC and AB, i.e. $y = 0$ and $z = 0$, form a harmonic pencil if

$$(gh' + g'h) = 0 \text{ ....}(1).$$

Now $\Delta = 2fgh$; $\Theta = 2f'gh + 2g'hf + 2h'fg$;

$$\Theta' = 2fg'h' + 2gh'f' + 2hf'g'; \text{ and } \Delta' = 2f'g'h'.$$

$$\therefore\ \Theta\Theta' - \Delta\Delta' = 4(gh' + g'h)(hf' + h'f)(fg' + f'g) = 0,$$

by equation (1).

If we take B and C as the circular points at infinity, the two conics become circles, and, if the tangents at one of their points of intersection A divide harmonically the lines joining A to the circular points at infinity, the circles cut orthogonally. The required condition that two circles cut orthogonally is thus $\Theta\Theta' = \Delta\Delta'$.

This may be easily verified by taking the equations to the circles in the form

$$x^2 + y^2 - a^2 = 0 \text{ and } x^2 + y^2 + 2gx + a^2 = 0.$$

281. Ex. 1. Shew that the relation $\Theta\Theta' = \Delta\Delta'$ is also satisfied when:

(1) the conics are a circle and a rectangular hyperbola, one of whose common chords is a diameter of the circle;

(2) the harmonic locus $\mathbf{F} = 0$ [Art. 113] degenerates into straight lines; and

(3) the harmonic envelope $\mathbf{F}' = 0$ [Art. 114] degenerates into a pair of points.

Ex. 2. Shew that the equations to two conics which satisfy the relation $\Theta\Theta' = \Delta\Delta'$ can always be reduced to the forms $x^2 + y^2 - z^2 = 0$ and $x^2 - y^2 + kz^2 = 0$.

282. *Shew that the condition that the pencil formed by joining any point on the conic $S = 0$ with its points of intersection with $S' = 0$ may be harmonic is*

$$2\Theta^3 - 9\Delta\Theta\Theta' + 27\Delta^2\Delta' = 0.$$

Let the conics be $\qquad L\alpha^2 + M\beta^2 + N\gamma^2 = 0 \text{ .... (1)},$

and $\qquad L_1\alpha^2 + M_1\beta^2 + N_1\gamma^2 = 0 \text{ .... (2)}.$

If, as in Art. 65, their points of intersection are P, Q, R, T, the equation to the lines PR and QT is

$$(LM_1 - L_1M)\beta^2 + (LN_1 - L_1N)\gamma^2 = 0.$$

By Page 128, Ex. 1, P, Q, R, T subtend a harmonic pencil at any point of (1) if PR, QT are conjugate with respect to it, and hence if (Art. 76)

$$NL(LM_1 - L_1M) + LM(N_1L - NL_1) = 0,$$

i.e. if $\qquad M_1NL + N_1LM = 2L_1MN.$

Now $\Delta = LMN$; $\Theta = L_1MN + M_1NL + N_1LM = 3L_1MN$ in this case;

$$\Theta' = LM_1N_1 + MN_1L_1 + NL_1M_1; \text{ and } \Delta' = L_1M_1N_1.$$

$$\therefore \Theta\Theta' - 3\Delta\Delta' = 3L_1^2MN(MN_1 + M_1N)$$

$$= \frac{6L_1^2MN}{L} \cdot L_1MN = \frac{2\Theta^3}{9\Delta}.$$

$$\therefore 2\Theta^3 - 9\Delta\Theta\Theta' + 27\Delta^2\Delta' = 0.$$

This is a homogeneous relation which is of the right dimensions according to the rule of Art. 217.

Aliter. Project the points Q, T into the circular points at infinity so that the conics become circles. Then P and R must subtend a right angle at any point of S, and hence PR must pass through the centre of S. Thus

$$S \equiv k(x^2+y^2-a^2),$$

and $$S' \equiv x^2+y^2-a^2+2\lambda(y-px), \text{ etc.}$$

233. *The equation* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ *represents a circle, the coordinates being areal; shew that its radius* ρ *is given by*

$$\rho^2=\frac{4R^2(abc+2fgh-af^2-bg^2-ch^2)}{(2f-b-c)(2g-c-a)(2h-a-b)},$$

where R *is the circum-radius of the triangle of reference.*

Take $$S=\lambda[ax^2+by^2+cz^2+2fyz+2gzx+2hxy]=0,$$

and $$S'=a_0^2yz+b_0^2zx+c_0^2xy=0,$$

where a_0, b_0, c_0 are the sides of the triangle of reference, and hence S' is the circum-circle.

Since $S=0$ represents a circle, λ can be chosen so that $S-S'=0$ has $x+y+z=0$ as a factor. Hence on substituting $x=-y-z$, we see that

$$\lambda[(a+b-2h)y^2+(a+c-2g)z^2+2yz(a+f-g-h)]$$
$$-[a_0^2yz-b_0^2z(y+z)-c_0^2y(y+z)]$$

is zero identically.

$$\therefore\ \lambda(a+b-2h)+c_0^2=0,$$
$$\lambda(a+c-2g)+b_0^2=0,$$

and $$\lambda(2a+2f-2g-2h)+(b_0^2+c_0^2-a_0^2)=0,$$

giving $$\lambda(b+c-2f)+a_0^2=0.$$

$$\therefore\ \lambda=\frac{a_0^2}{2f-b-c}=\frac{b_0^2}{2g-c-a}=\frac{c_0^2}{2h-a-b}\ \ldots\ldots\ldots\ldots(1).$$

Hence $$\Delta=\lambda^3[abc+2fgh-af^2-bg^2-ch^2],$$

and $$\Delta'=\tfrac{1}{4}a_0^2b_0^2c_0^2.$$

Since $S=0$ is now in the proper form, we have, by Art. 218,

$$\frac{\rho^2}{R^2}=\frac{\Delta}{\Delta'}=\frac{4(abc+2fgh-af^2-bg^2-ch^2)}{(2f-b-c)(2g-c-a)(2h-a-b)},$$

on substituting for a_0^2, b_0^2, c_0^2 from (1).

This is the result with **Areal** coordinates. If we use **Trilinear** coordinates we should have similarly

$$\frac{\rho^2}{R^2}=\frac{4a_0^2b_0^2c_0^2[abc+2fgh-af^2-bg^2-ch^2]}{(2fb_0c_0-bc_0^2-cb_0^2)(2gc_0a_0-ca_0^2-ac_0^2)(2ha_0b_0-ab_0^2-ba_0^2)}.$$

234. *If* $S=0$ *and* $S'=0$ *are the equations to two conics,* $\mathbf{F}=0$ *their harmonic tangent locus, and* $\mathbf{F}'=0$ *their harmonic envelope, to prove that* $\mathbf{F}'=\Theta'S+\Theta S'-\mathbf{F}$, *where the forms used for* $\mathbf{F}$ *and* $\mathbf{F}'$ *are such that their discriminants are*

$$\Delta\Delta'(\Theta\Theta'-\Delta\Delta') \text{ and } (\Theta\Theta-\Delta\Delta')^2.$$

Let the two conics be referred to their common self-conjugate triangle, so that

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$

and $$S' \equiv a'x^2 + b'y^2 + c'z^2 = 0.$$

Hence, by Arts. 113 and 114,

$$\mathbf{F} = aa'(bc' + b'c)x^2 + bb'(ca' + c'a)y^2 + cc'(ab' + a'b)z^2 = 0,$$

and

$$\mathbf{F}' = (ab' + a'b)(ca' + c'a)x^2 + (bc' + b'c)(ab' + a'b)y^2 + (ca' + c'a)(bc' + b'c)z^2 = 0.$$

Now

$$\Theta = a'bc + b'ca + c'ab, \text{ and } \Theta' = ab'c' + bc'a' + ca'b'.$$

Hence the coefficient of x^2 in

$$\begin{aligned}\Theta'S + \Theta S' - \mathbf{F} &= a(ab'c' + bc'a' + ca'b') \\ &\quad + a'(a'bc + b'ca + c'ab) - aa'(bc' + b'c) \\ &= a^2b'c' + a'^2bc + caa'b' + aba'c' \\ &= (ab' + a'b)(ca' + c'a),\end{aligned}$$

and so for the coefficients of y^2 and z^2.

Hence $$\mathbf{F}' = \Theta'S + \Theta S' - \mathbf{F}.$$

The discriminant of $\mathbf{F}$

$$= abca'b'c'(bc' + b'c)(ca' + c'a)(ab' + a'b).$$

Also

$$\begin{aligned}\Theta\Theta' - \Delta\Delta' &= (a'bc + b'ca + c'ab)(ab'c' + bc'a' + ca'b') - abc \,.\, a'b'c' \\ &= (bc' + b'c)(ca' + c'a)(ab' + a'b).\end{aligned}$$

Hence the discriminant of $\mathbf{F}$

$$= \Delta\Delta'[\Theta\Theta' - \Delta\Delta'].$$

Also the discriminant of $\mathbf{F}'$

$$= (bc' + b'c)^2(ca' + c'a)^2(ab' + a'b)^2 = [\Theta\Theta' - \Delta\Delta']^2.$$

Hence the forms chosen for $\mathbf{F}$, $\mathbf{F}'$ are as stated.

Cor. If $\Theta\Theta' - \Delta\Delta' = 0$, the harmonic locus breaks into two straight lines, and the harmonic envelope also breaks into factors, and gives two points only. This is the case, as in Exs. 1 and 2, p. 11, when the two conics are orthogonal circles.

235. Such a function as $\mathbf{F}'$ is said to be a covariant of S, S' and $\mathbf{F}$.

Any function ϕ is a covariant of other functions $S, S', \ldots$ when it is derived by some rule from those functions, and when, if the variables involved are transformed by any linear substitution, the result obtained by transforming ϕ differs only by at most a constant multiplier from that which we should get by first transforming $S, S', \ldots$, and then from them deriving ϕ by the same rule as before.

236. In a similar manner we can shew that if Σ and Σ' be the tangential equations of two conics, and Φ and Φ' the tangential equations of their harmonic locus and harmonic envelope respectively, then

$$\Phi' = \Theta'\Sigma + \Theta\Sigma' - \Phi.$$

For $$\Sigma = l^2bc + m^2ca + n^2ab;$$

$$\Sigma' = l^2b'c' + m^2c'a' + n^2a'b';$$

$$\Phi = bcb'c'(ca' + c'a)(ab' + a'b)l^2 + \ldots + \ldots;$$

and $$\Phi' = aa'bb'cc'[(bc' + b'c)l^2 + \ldots + \ldots].$$

$\Theta =$ the invariant derived from Σ and $\Sigma' = abc(ab'c' + \ldots + \ldots)$, and Θ' similarly $= a'b'c'(a'bc + \ldots + \ldots)$.

[It may be verified, as in Art. 234, that the forms of Φ and Φ' are so chosen that their discriminants are respectively $(\Theta\Theta' - \Delta\Delta')^2$ and $\Delta\Delta'(\Theta\Theta' - \Delta\Delta')$, where Δ, Θ, Θ', and Δ' belong to the system Σ and Σ'.]

Just as in Art. 234, it may now be shewn that

$$\Phi' = \Theta'\Sigma + \Theta\Sigma' - \Phi.$$

237. *If S_1 be the polar reciprocal of S with respect to S', and S_1' the polar reciprocal of S' with regard to S, then*

$$S_1 = \Theta S' - \mathbf{F} = \mathbf{F}' - \Theta' S,$$

and $$S_1' = \Theta' S - \mathbf{F} = \mathbf{F}' - \Theta S'.$$

With the notation as in Art. 234, we have, from Art. 178, that

$$S_1 = a'^2bcx^2 + b'^2cay^2 + c'^2abz^2 = 0,$$

and $$S_1' = a^2b'c'x^2 + b^2c'a'y^2 + c^2a'b'z^2 = 0.$$

Now

$$\begin{aligned}\Theta S' - \mathbf{F} &= (a'bc + b'ca + c'ab)(a'x^2 + b'y^2 + c'z^2) \\ &\quad - aa'(bc' + b'c)x^2 - bb'(ca' + c'a)y^2 - cc'(ab' + a'b)z^2 \\ &= a'^2bcx^2 + b'^2ca\,y^2 + c'^2ab\,z^2 = S_1.\end{aligned}$$

$\therefore\ S_1 = \Theta S' - \mathbf{F} = \mathbf{F}' - \Theta' S$, by Art. 234.

Also

$$\begin{aligned}\Theta' S - \mathbf{F} &= (ab'c' + bc'a' + ca'b')(ax^2 + by^2 + cz^2) \\ &\quad - aa'(bc' + b'c)x^2 - bb'(ca' + c'a)y^2 - cc'(ab' + a'b)z^2 \\ &= a^2b'c'x^2 + b^2c'a'y^2 + c^2a'b'z^2 = S_1'.\end{aligned}$$

$\therefore\ S_1' = \Theta' S - \mathbf{F} = \mathbf{F}' - \Theta S'$, by Art. 234.

Cor. If the conics S and S' are such that $\Theta = 0$, then the harmonic locus $\mathbf{F}$ is the polar reciprocal of S with respect to S', and the harmonic envelope $\mathbf{F}'$ is the polar reciprocal of S' with respect to S.

If the conics are such that $\Theta' = 0$, then the harmonic locus $\mathbf{F}$ is the polar reciprocal of S' with respect to S, and the harmonic envelope $\mathbf{F}'$ is the polar reciprocal of S with respect to S'.

238. That there must be a linear relation between S_1, S' and $\mathbf{F}$ follows from the fact that the points of contact with S' of any common tangents to S and S' are the poles of tangents to S with regard to S', and are thus points on the polar reciprocal S_1, and through these points passes $\mathbf{F}$, by Art. 117. Hence we must have a relation of the form

$$S_1 = lS' + m\mathbf{F}.$$

239. *Shew that the general equation of a conic touching the four common tangents of two conics $S=0$ and $S'=0$ is $\Delta S + k\mathbf{F} + k^2\Delta' S' = 0$, where k is a variable parameter and $\mathbf{F}=0$ is the harmonic locus.*

Let $\qquad S \equiv ax^2 + by^2 + cz^2 = 0,$

and $\qquad S' \equiv a'x^2 + b'y^2 + c'z^2 = 0.$

Let Σ and Σ' be the tangential equations to these conics, so that, by Art. 72,

$$\Sigma = l^2bc + m^2ca + n^2ab = 0, \text{ and } \Sigma' = l^2b'c' + m^2c'a' + n^2a'b' = 0.$$

The tangential equation of any conic touching the four common tangents of S and S' is

$$\Sigma + k\Sigma' = 0,$$

i.e. $\qquad (bc + kb'c')l^2 + (ca + kc'a')m^2 + (ab + ka'b')n^2 = 0.$

For the point equation we have, by Art. 73, the coefficient of x^2

$$=(ca+kc'a')(ab+ka'b')=a^2bc+kaa'(bc'+b'c)+k^2a'^2b'c',$$

and so for the coefficients of y^2 and z^2. The required equation is thus

$$\begin{aligned}&abc\,[ax^2+by^2+cz^2]\\&\qquad+k\,[aa'(bc'+b'c)x^2+bb'(ca'+c'a)y^2+cc'(ab'+a'b)z^2]\\&\qquad+k^2a'b'c'(a'x^2+b'y^2+c'z^2)=0,\end{aligned}$$

i.e.

$$\Delta S+k\mathbf{F}+k^2\Delta'S'=0.$$

Cor. The envelope of these conics is the four common tangents themselves. Hence the equation to these four common tangents is $\mathbf{F}^2=4\Delta\Delta'SS'$. [Cf. Art. 117.]

240. *Shew that the tangential equation of* $S+kS'=0$ *is*

$$\Sigma+k\Phi+k^2\Sigma'=0,$$

and interpret the equations $\Phi=0$ *and* $\Phi^2-4\Sigma\Sigma'=0$. *Shew that the point equation of* $\Sigma+k\Phi=0$ *is*

$$\Delta S+k\,[\Theta S+\Delta S']+k^2[\Theta'S+\Theta S'-\mathbf{F}]=0,$$

and interpret the equation

$$[\Theta S+\Delta S']^2-4\Delta S\,[\Theta'S+\Theta S'-\mathbf{F}]=0.$$

Let $S\equiv ax^2+by^2+cz^2=0$, and $S'\equiv a'x^2+b'y^2+c'z^2=0$.

Then the tangential equation of

$$(a+ka')x^2+(b+kb')y^2+(c+kc')z^2=0$$

is

$$l^2[b+kb'][c+kc']+\ldots+\ldots=0,$$

i.e.

$$(bcl^2+\ldots+\ldots)+k\,[l^2(bc'+b'c)+\ldots+\ldots]+k^2[l^2b'c'+\ldots+\ldots]=0,$$

i.e.

$$\Sigma+k\Phi+k^2\Sigma'=0,$$

where Φ is the tangential equation of the harmonic envelope. The equation $\Phi^2=4\Sigma\Sigma'$ is the tangential equation of the envelope of the above conics, *i.e.* it is the tangential equation of the four common points of the conics S, S'. [Cf. Art. 116.]

The equation

$$\Sigma+k\Phi=0$$

is

$$\{bc+k(bc'+b'c)\}\,l^2+\ldots+\ldots=0.$$

For the point equation the coefficient of x^2

$$\begin{aligned}&=[ca+k(ca'+c'a)][ab+k(ab'+a'b)]\\&=a^2bc+k\,[a(a'bc+b'ca+c'ab)+a'.abc]+k^2(ca'+c'a)(ab'+a'b).\end{aligned}$$

Hence the point equation is

$$\begin{aligned}&abc\,[ax^2+by^2+cz^2]\\&\qquad+k\,[(ax^2+by^2+cz^2)(a'bc+b'ca+c'ab)+abc\,(a'x^2+b'y^2+c'z^2)]\\&\qquad+k^2[(ca'+c'a)(ab'+a'b)x^2+\ldots+\ldots]=0,\end{aligned}$$

i.e.

$$\Delta S+k\,[\Theta S+\Delta S']+k^2\mathbf{F}'=0,$$

i.e.

$$\Delta S+k\,[\Theta S+\Delta S']+k^2[\Theta'S+\Theta S'-\mathbf{F}]=0,$$

by Art, 234.

The equation

$$[\Theta S' + \Delta S']^2 - 4\Delta S[\Theta' S + \Theta S' - \mathbf{F}] = 0$$

is the envelope of these conics, *i.e.* it is the equation to the four common tangents to S and $\mathbf{F}'$.

241. Since the results of the last two articles only contain quantities which are invariants with respect to S and S', or which, like $\mathbf{F}=0$ and $\mathbf{F}'=0$, are defined by a geometrical property which is independent of the particular axes of coordinates or triangles of reference which are used, these results will remain true whatever be the particular forms that are used for S and S', and, in particular, will hold when S and S' are given by the general equations of the second degree.

This can be verified, as in the previous articles, by taking the general equations for S and S' and using for F and Φ the values given in Art. 115.

EXAMPLES.

1. Determine the homogeneous relations which exist between the invariants of the conics

$$S_1 \equiv (\alpha x + \beta y)^2 - 2x = 0;$$

$$S_2 \equiv \beta^2 (x^2 + y^2) - 2x = 0;$$

and $$S_3 \equiv 2xy + 2\beta\lambda x + 4\alpha\lambda y - \frac{2\lambda}{\beta} = 0;$$

and verify the corresponding geometrical relations between the conics.

2. Find what homogeneous relations exist between the invariants in the two following cases:

(1) S is a parabola, and S' a rectangular hyperbola whose centre is at the focus of S;

(2) S is a circle, and S' a parabola whose focus is on S.

3. From the fact that, if a conic touches the locus of the feet of the normals drawn to it from any point P, then the locus of P is the evolute of the conic, obtain the evolutes of the parabola $y^2 = 4px$ and of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

4. If $S'=0$ be a given conic, the locus of the centre of a variable conic $S=0$, which is self-conjugate with regard to a given triangle, and such that the invariant Θ of S and S' vanish, is a straight line.

5. Two conics S and S' intersect in A; the tangents to S and S' at A meet S' and S in B and C respectively and BC meets the conics again in B', C'; if B' and C' are the harmonic conjugates of B and C, prove that $\Theta\Theta' + \Delta\Delta' = 0$.

6. Shew that a conic can be drawn to have contact of the third order with the conics $S=0$ and $S'=0$ if $\Delta\Theta'^3 = \Delta'\Theta^3$.

7. $F=0$ and $F'=0$ are the harmonic locus and harmonic envelope of two conics $S=0$ and $S'=0$. If triangles can be inscribed in F, which are self-conjugate with respect to F', then

$$\Theta\Theta'+3\Delta\Delta'=0.$$

8. If $F=0$ and $F'=0$ have double contact, then $S=0$ and $S'=0$ have double contact, or else $\Delta\Theta'^2=\Delta'\Theta^2$.

9. If the four points of contact with $S=0$ of the common tangents to $S=0$ and $S'=0$ are joined to any point of S, and the lines so formed give a harmonic pencil, prove that

$$2\Theta'^3-9\Theta\Theta'\Delta'+27\Delta\Delta'^2=0.$$

10. If one pair of common chords of two conics are conjugate lines with regard to each of the conics, prove that $\Theta\Theta'=9\Delta\Delta'$.

11. Two conics $S=0$ and $S'=0$ touch; shew that the ratio of their curvatures at the point of contact is equal to the ratio of the unequal roots of the equation

$$\Delta k^3+\Theta k^2+\Theta' k+\Delta'=0.$$

12. An infinite number of quadrilaterals can be inscribed in S and circumscribed about S'. If S_1 is the polar reciprocal of S with respect to S', shew that an infinite number of triangles can be inscribed in S and circumscribed about S_1.

13. *Find the locus of the ortho-centre of a triangle which is inscribed in a conic S and circumscribed to a conic S'.*

The principal axes of S' being the axes of coordinates, we have

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

and

$$S'\equiv a'x^2+b'y^2-1=0.$$

Let (x', y') be the ortho-centre of any such triangle and d the radius of its self-conjugate circle, so that the equation to the latter is

$$S''\equiv(x-x')^2+(y-y')^2-d^2=0.$$

By Arts. 221 and 219 since the triangle is inscribed in S, circumscribed to S', and self-conjugate to S'', the Θ' of S and S'' is zero, and also the Θ of S' and S'' is zero.

Hence $$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c-(a+b)\,d^2=0,$$

and

$$a'+b'-(x'^2+y'^2-d^2)\,a'b'=0.$$

Hence, on eliminating d^2, the required locus is the conic

$$S=(a+b)\left(x^2+y^2-\frac{1}{a'}-\frac{1}{b'}\right).$$

14. *A triangle ABC is self-conjugate to a conic S, and two of its angular points, B and C, lie on a conic S'; shew that the locus of the third angular point A is $\Theta S-\Delta S'=0$. Prove also that the side BC touches the harmonic envelope of S and S', and that the sides AB, AC touch the reciprocal polar of S' with regard to S.*

Taking ABC as the triangle of reference, we have

$$S \equiv ax^2+by^2+cz^2=0,$$

and

$$S' \equiv a'x^2+2f'yz+2g'zx+2h'xy=0.$$

Clearly the conic $a'S-aS'=0$ goes through the point A.

But $\Delta=abc$, and $\Theta=a'bc$. $\therefore \dfrac{a'}{\Theta}=\dfrac{a}{\Delta}$, and hence $\Theta S-\Delta S'=0$ is a conic through A. But since this equation contains only S, S' and the invariants, it is clear that it remains the same whatever triangle of reference is taken. Hence the conic is a fixed one. It follows, as would be expected from Art. 219, that the locus reduces to $S'=0$ when Θ is zero.

The tangential equation to the harmonic envelope is (Art. 115)

$$\Phi=ca'm^2+a'bn^2-2af'mn-2bg'nl-2ch'lm=0.$$

This is satisfied by $m=0$, $n=0$, *i.e.* the base BC always touches the invariant conic $\mathbf{F}'=0$.

As in Art. 179 the polar reciprocal of S' with regard to S is

$$a^2f'^2x^2+b^2g'^2y^2+c^2h'^2z^2-2bc\,(g'h'-a'f')\,yz$$
$$-2cah'f'zx-2abf'g'xy=0.$$

This touches both AB and AC. But, since the equation of this polar reciprocal can be put into the form $\Theta'S-F=0$, it is an invariant conic. Hence the sides AB, AC always touch it.

15. Triangles can be inscribed in a conic $S=0$ which are self-conjugate for a conic $S'=0$. Shew that the triangle formed by the tangents at the angular points of any such triangle is inscribed in the conic $\Delta S'-\Theta S=0$.

16. Two conics, S and S', are such that triangles can be inscribed in S' whose sides touch S. Shew that the locus of the point of intersection of the lines joining the vertices of any such triangle to the point of contact of opposite sides is the conic $3\Delta S'-2\Theta S=0$.

17. A triangle is inscribed in a conic $S=0$ and two of its sides touch the conic $S'=0$; shew that the envelope of the base is the conic

$$[\Theta'^2-4\Delta'\Theta]\,S+4\Delta\Delta'S'=0.$$

18. A triangle circumscribes a conic $S=0$ and the extremities of its base lie on the conic $S'=0$; shew that the vertex lies on the conic

$$\{\Theta^2-4\Delta\Theta'\}^2\,S+16\Delta^3\Delta'\,S'+4\Delta\,\{\Theta^2-4\Delta\Theta']\,\mathbf{F}=0.$$

[When $\Theta^2=4\Delta\Theta'$, the locus reduces to $S'=0$. (Art. 222.)]

19. The conic $S=0$ touches three sides of a quadrilateral, and the conic $S'=0$ passes through the four angular points; prove that the envelope of the fourth side is the conic

$$\{\Theta^2-4\Delta\Theta'\}^2\,S+8\Delta\,[\Theta^3-4\Delta\Theta\Theta'+8\Delta^2\Delta']\,S'=0.$$

ANSWERS

(Pages 58, 59.)

2. $\{(mn+1)(\alpha m+\beta n)+\gamma(m^3+n^3)\}\{(mn-1)(\alpha m-\beta n)+\gamma(m^3-n^3)\}=0.$
6. A conic touching two of the fixed lines.
7. A conic circumscribing the triangle of reference.
13. A cubic circumscribing the triangle of reference.

(Pages 77, 78.)

2. $(\beta\cos B-\gamma\cos C)^2=\alpha\sin A\,(\beta\sin B+\gamma\sin C)$;
   $c\cos^2 B\,.\,\beta+b\cos^2 C\,.\,\gamma=0.$
12. The nine-point circle.

(Pages 85–87.)

24. $x^2+y^2+z^2-2yz-2zx-2xy=0.$

(Pages 92–94.)

13. (1). $(1,0)$, $(-1,0)$ and $(0,1)$; $(x+y-1)^2+(x-y+1)^2-4y^2=0$;
    $(x+y-1)^2-(x-y+1)^2+y^2=0.$
    (2). $(1,0)$, $(-1,0)$ and $(-\frac{1}{7},\frac{4}{7})$;
    $4(x+2y-1)^2-(2x-3y+2)^2-3y^2=0$;
    $4(x+2y-1)^2+(2x-3y+2)^2-5y^2=0.$
    (3). $(\frac{1}{2},-\frac{1}{2})$, $(-\frac{1}{3},\frac{1}{3})$ and $(-2,-3)$;
    $(2x-y+1)^2-(x-y-1)^2+(x+y)^2=0$;
    $-(2x-y+1)^2+2(x-y-1)^2+2(x+y)^2=0.$
14. $X\equiv y+z=0$; $Y\equiv z+x=0$; $Z\equiv x+y=0$; $X^2-Y^2+Z^2=0$;
    $X^2+Y^2-2Z^2=0.$

(Page 106.)

10. $x^2=4yz.$

(Pages 111–114.)

3. $x=0$; $y=0$; $x+2z=0$; $x-12y-4z=0.$

(Page 144.)

$$\frac{x}{x'}=\frac{y}{3x'-6y'}=\frac{1}{x'-3y'+3}.$$

(Page 149.)

1. $56l^2+9m^2-15n^2-42mn-52nl+44lm=0$.
3. $(\frac{2}{3}, \frac{4}{3})$; $x+y=2$.

(Pages 159, 160.)

3. The conic $ayz+bzx+cxy=0$.
5. $(3, -3, -2)$; $x+y=0$ and $17x+5y+18z=0$.
6. $\left(0, \frac{1}{M}, \frac{1}{N}\right)$, etc.; $(M+N, N+L, L+M)$.

(Page 168.)

1. $(2, 1)$; $(-2, -1)$.

(Page 224.)

1. $\Theta\Theta'=9\Delta\Delta'$; $\Theta^2=4\Delta\Theta'$; $\Theta=0$.
2. $\Theta=0$; $\Theta'^2=4\Delta'\Theta$.

BY THE SAME AUTHOR

A TREATISE ON ELEMENTARY DYNAMICS. Crown 8vo. Tenth Edition. 8*s.*

SOLUTIONS OF THE EXAMPLES IN THE ELEMENTARY DYNAMICS. Cr. 8vo. 8*s.* 6*d.*

THE ELEMENTS OF STATICS AND DYNAMICS. Ex. Fcp. 8vo.

Part I. Elements of Statics. Seventeenth Edition. 5*s.* 6*d.*
Part II. Elements of Dynamics. Seventeenth Edition. 5*s.*
The two Parts bound in one volume, 10*s.*

SOLUTIONS OF THE EXAMPLES IN THE ELEMENTS OF STATICS AND DYNAMICS. Ex. Fcp. 8vo. 9*s.*

MECHANICS AND HYDROSTATICS FOR BEGINNERS. Ex. Fcp. 8vo. Eighteenth Edition. 5*s.*

PLANE TRIGONOMETRY. Crown 8vo. 10*s.* Or in two Parts.

Part I. Up to and including the Solution of Triangles. Fifteenth Edition. 6*s.*
Part II. De Moivre's Theorem and the Higher Portions. Twelfth Edition. 5*s.*

SOLUTIONS OF THE EXAMPLES IN THE PLANE TRIGONOMETRY. Crown 8vo. In two Parts. Part I, 15*s.*; Part II, 7*s.* 6*d.*

THE ELEMENTS OF TRIGONOMETRY WITH FOUR-FIGURE LOGARITHM TABLES. Ex. Fcp. 8vo. Fifth Edition. 4*s.* 6*d.*

THE ELEMENTS OF HYDROSTATICS, being a Companion Volume to the Elements of Statics and Dynamics. Ex. Fcp. 8vo. Seventh Edition. 5*s.* 6*d.*

SOLUTIONS OF THE EXAMPLES IN THE ELEMENTS OF HYDROSTATICS. Ex. Fcp. 8vo. 6*s.*

DYNAMICS OF A PARTICLE AND OF RIGID BODIES. Demy 8vo. Third Edition. 14*s.*

STATICS. Demy 8vo. Third Edition. 14*s.*

CAMBRIDGE UNIVERSITY PRESS
C. F. CLAY, Manager
London: FETTER LANE, E.C. 4

BY THE SAME AUTHOR

COORDINATE GEOMETRY. Crown 8vo. Twenty-first Impression. 7*s*. The Straight Line and Circle only, 4*s*.

SOLUTIONS OF THE EXAMPLES IN THE COORDINATE GEOMETRY (with A. S. GOSSET-TANNER, M.A.). 10*s*. net.

AN ARITHMETIC FOR SCHOOLS. Sixteenth Impression. Globe 8vo. With or without Answers, 5*s*. Or in two Parts, with Answers, 3*s*. each. The Examples alone, 3*s*. 6*d*. The Answers alone, 6*d*.

A NEW EDITION OF DR TODHUNTER'S ALGEBRA FOR BEGINNERS. Globe 8vo. Twelfth Impression. 4*s*. without Answers. 5*s*. with Answers. Answers only, 1*s*. 3*d*. **KEY**, 10*s*. 6*d*. net.

A NEW EDITION OF DR TODHUNTER'S EUCLID. Globe 8vo. Fourth Impression. 5*s*. Also Book I., 1*s*. 3*d*. Books I. and II., 1*s*. 9*d*. Books I.—IV., 3*s*. 6*d*.

With L. W. GRENVILLE, M.A.

A SHILLING ARITHMETIC. Seventeenth Impression. Globe 8vo. 1*s*. 6*d*. With Answers, 2*s*.

London: MACMILLAN AND CO., LIMITED.